A posteriori error estimate for finite volume approximations of nonlinear heat equations

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Abstract

In this contribution we derive a posteriori error estimate for finite volume approximations of nonlinear convection diffusion equations in the $L^\infty(L^1)$-norm. The problem is discretized implicitly in time by the method of characteristics, and in space by piecewise constant finite volume methods. The analysis is based on a reformulation for finite volume methods. The derived a posteriori error estimates are based on the Kruzkov technique. The idea exists of a reformulation into finite volume methods, such that the estimates, that are known in the context of the finite element methods, can be reused. Our error estimates have the correct convergence order and recover the standard a posteriori error estimates known for parabolic equations.

Keywords: finite volume scheme, a posterior error estimate, convection-diffusion equation, characteristic method, parabolic equation.

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1 Introduction and Mathematical Model

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The motivation to study the nonlinear heat equation came from the technical application, for example simulation of sublimation for a crystal growth apparatus, or simulations for thermodynamical chemical reactors, see [11], [12] and [14].

All this models consist of nonlinear transport terms. It is delicate to apply a standard finite element method because of their stability and consistency problems, see [1] and [2]. Further the high computation time for discretizations with uniform grids needs to apply adaptive methods, that are based on a posteriori error estimates, see [4], [5], [10] and [9]. Based on the work of [5], a posteriori error estimates are developed for finite element methods. In our applications we deal with mass conservation and impulse conservation, and therefore the usage of conservative discretization methods are important [6]. Our underlying methods are finite volume methods, which are conservative discretizations and can be treated as a mixture between finite difference and finite element methods, see [6], [8] and [13].

For the finite volume methods a huge class of a posteriori error estimates are developed for linear parabolic equations, see [6] and [9], and are applied in software packages as [7]. On the other hand for nonlinear problems, for example for the nonlinear diffusion term, works are done in the direction of finite element methods, confer [2] and [5].

Therefore the motivation is to generalize the results of a posteriori error estimates for the finite element methods to the finite volume methods.

In the next section we describe the used discretization.

2 Model Equation

For the definition of the finite volume scheme and the following a posteriori error analysis, we will focus on the following model problem.

Let $\Omega$ be a bounded domain $\mathbb{R}^d (d = 1, 2, 3)$ with Lipschitz boundary and $T > 0$. 
We consider the following equation:

\[
\begin{align*}
\partial_t u + \text{div} f(u) - \Delta A(u) &= g, \text{ in } \Omega \times (0,T], \\
u(0,\cdot) &= u_0, \text{ in } \Omega, \\
u &= 0, \text{ on } \partial \Omega \times (0,T),
\end{align*}
\]

where \( u = u(x,t) \in \mathbb{R} \), with \((x,t) \in \Omega_T = \Omega \times [0,T]\). We assume, that our function \( f : \mathbb{R} \to \mathbb{R}^d \) is locally Lipschitz continuous, the function \( A : \mathbb{R} \to \mathbb{R} \) is non-decreasing and locally Lipschitz continuous, \( g \in L^\infty(\Omega_T) \), and \( u_0 \in L^\infty(\Omega) \).

In the following, we use the assumptions (H1)-(H3) [2],

(H1) \( f : \mathbb{R} \to \mathbb{R}^d \) is locally Lipschitz continuous, 
\[ f(0) = 0; f' \] is uniformly Lipschitz continuous in \( \mathbb{R} \).

(H2) \( A : \mathbb{R} \to \mathbb{R} \) is locally Lipschitz continuous, \( A(0) = 0; A'(s) > 0 \) 
for any \( s \in \mathbb{R} \), and \( A' \cdot A^{-1} \) is globally Hölder continuous in \( \mathbb{R} \), i.e. 
\[ \left| (A' \cdot A^{-1})(s) - (A' \cdot A^{-1})(r) \right| \leq C \left| s - r \right| ^\gamma \quad \forall s, r \in \mathbb{R}, \]
for some constant \( 0 < \gamma \leq 1 \) and some constant \( C > 0 \).

(H3) \( g \in L^\infty(Q) \), \( u_0 \in L^\infty(\Omega) \).

3 Discretization

In this section, we apply the characteristic finite difference method for the convection term and the cell-centered finite volume method for the diffusion term.

Because of the splitting of the two operators, we can analyze the two methods independently. Therefore the estimation can also be discussed in two parts, namely the error estimates for the convection part and the diffusion part.

3.1 Characteristic Finite Difference Method for the Convection Term

We deal with the following equation:

\[
\partial_t u + \nabla \cdot vu = 0.
\]
The initial conditions are given and $u(t, \gamma)$ is explicitly given for $t > 0$ at the inflow boundary $\gamma \in \partial^{\text{in}} \Omega$.

The characteristic finite difference is given in the following description.

The exact solution of (4) can directly be defined using the so-called forward tracking form of characteristic curves. If the solution of (4) is known at some time point $t_0 \geq 0$ and some point $y \in \Omega \cup \partial^{\text{in}} \Omega$, then $u$ remains constant for $t \geq t_0$ along the characteristic curve $X = X(t)$, i.e. $u(t, X(t)) = u(t_0, y)$ and

$$X(t) = X(t; t_0, y) = y + \int_{t_0}^{t} \frac{\vec{v}(X(s))}{X(s) \phi(X(s))} ds. \quad (5)$$

The characteristic curve $X(t)$ starts at the time $t = t_0$ in the point $y$, i.e. $X(t_0; t_0, y) = y$, and it is tracked forward in time for $t > t_0$. Of course, we can obtain, that $X(t) \notin \Omega$, i.e. the characteristic curve can leave the domain $\Omega$ through $\partial^{\text{out}} \Omega$.

### 3.2 Cell-Centered Finite Volume Discretization for the Diffusion Term

In order to define the cell-centered finite volume discretization of (1), we now define some notation for the underlying partition of $\Omega \times (0, T]$.

Let $J := \{t_0, ..., t_N\}$ be a partition of $[0, T]$ and $\Delta t^n := t^{n+1} - t^n$ the step size of $J$. Furthermore let $T = \{T_j | j \in I\}$ be a regular partition of $\Omega$ such that the common interface $S_{jl}$ of two neighboring cells $T_j$ and $T_l$ is included in a hyperplane.

The index set of neighboring cells to $T_j$ is denoted by $N(j)$. Corresponding to an edge $S_{jl}$, we consider the outward normal vector $\mathbf{n}_{jl}$.

The oriented set of faces $E$ is defined as $E := \{(j, l)|S_{jl}$ is an edge of $T$ and $j > l\}$. We use the concept of ghost cells to compute the flux at the boundary. Let the index set $I_{\text{ext}}$ be such that $I^n_{\text{ext}} \cap I = \emptyset$ and such that for each edge $S \subset \partial \Omega$ there exists a unique pair of indices $(j, l) \in I \times I_{\text{ext}}$ with $\partial T_j \cap S = S$. In this situation we denote $S_{jl} := S$. Accordingly, the set of edges located on the boundary of $\Omega$ is denoted by

$$S^n_{\text{ext}} := \{(j, l) \in I \times I_{\text{ext}}| S_{jl}$ is an exterior edge of $T^n\}.$$
The mesh $T$ is assumed to be such that there exists a family of points \( \{x_j | j \in I\} \) with $x_j \in T_j$ for all $j \in I$ and the straight line $(x_j, x_l)$ is perpendicular to $S_{jl}$. Let us denote by $z_{jl}$ the intersection point of the line $(x_j, x_l)$ with $S_{jl}$ and define $\delta_{jl} := |x_j - z_{jl}|$, $d_{jl} := \delta_{jl} + \delta_{lj} = |x_j - x_l|$. Finally we define $h_j := \text{diam} (T_j)$, $h_{jl} := \text{diam} (T_j \cup T_l)$, and $h_{\text{min}} := \min_{j \in I} h_j$ (see Figure 1 for an illustration).

Assumption 3.1 We assume that there exists an $\alpha > 0$, such that we have for all $j \in I, l \in N(j)$: $\alpha h_j^d \leq |T_j|$, $\alpha |\partial T_j| \leq h_j^{d-1}$, and $\alpha h_j \leq d_{jl}$, where $|\cdot|$ denotes the Hausdorff measure.

Remark 3.2 Assumption 3.1 is for instance satisfied by Voronoi meshes based on Delaunay triangulations. For other choices and a possible weakening of this condition see [6].

The cell-centered upwind finite volume scheme for computing approximate solutions to (1) is defined as follows.

$$u_j^0 := \frac{1}{|T_j|} \int_{T_j} u_0, \quad j \in I, \quad u_l^n := 0, \quad j \in I_{\text{ext}}, \quad g_j^n := \frac{1}{|T_j|} \int_{T_j} g(t^{n+1}),$$

$$u_j^{n+1} - u_j^n + \frac{\Delta t^n}{d_{jl}} \sum_{l \in N(j)} \frac{A(u_l^{n+1}) - A(u_j^{n+1})}{|T_j|} = \Delta t^ng_j^{n+1},$$

for all $n \in \{0, \ldots, N\}$ and $j, l \in I^n$.

4 Definition of $u_h$ and Reformulation of the Finite Volume Scheme

The goal of the following reformulation of the finite volume scheme is to paste the method into a variational formulation and then proceed for the a posteriori error analysis along the lines of Chen and Ji [2], where optimal a posteriori error estimates were derived for a characteristic finite element discretization of a nonlinear convection-diffusion equation. The following definition of the discrete solution $u_h : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$
and the corresponding reformulation of the finite volume method is one-to-one from the work of Herbin and Ohlberger [9] and Ohlberger [15].

Let us denote the set of vertices of the partition $\mathcal{T}$ by $\mathcal{V} := \{p_k | k \in K\}$ and the index set of vertices of the face $S_{jl}$ by $V(jl) \subset K$. In addition let $i := 0, \ldots, n-1$ with $n := |V(jl)|$ denote a local numbering, such that with a bijection $\alpha$ we have $p_i = p_{\alpha(i)}$, $\alpha(i) \in V(jl)$, and $p_i, p_{S^1(i)}$ have a joint edge in $\tilde{S}_{jl}$, where $S^m(i) := (i+m) \mod n$. For $i := 0, \ldots, n-1$, define the d-simplex $\tilde{T}_{\alpha(i)}$ as the convex hull of the points $x_j, x_l, p_i, \ldots, p_{S^{d-2}(i)}$. Then the dual triangulation $\tilde{T}$ to $\mathcal{T}$ is given by $\tilde{T} := \{\tilde{T}_{jl}^k | (j, l) \in E, k \in V(jl)\}$. Furthermore we define for each $p_k \in \mathcal{V}$ the Voronoi cell $\Omega_k$ by $\Omega_k := \bigcup_{(j,l) \in E | S_{jl} \cap p_k \neq \emptyset} \tilde{T}_{jl}^k$ and for each pair $(j, l) \in E$ the diamond cell $\mathcal{D}_{jl}$ by $\mathcal{D}_{jl} := \bigcup_{k \in V(jl)} \tilde{T}_{jl}^k$. Furthermore, for each simplex $\tilde{T}_{jl}^k$, $k = \alpha(i)$ we consider the local coordinate system given by the outward normal vector $\mathbf{n}_{jl}$ and the tangent vectors $\mathbf{t}_{jl}^0, \cdots, \mathbf{t}_{jl}^{d-2}$, defined as

$$\mathbf{t}_{jl}^m := \frac{p_{S^m(i)} - z_{jl}}{|p_{S^m(i)} - z_{jl}|}.$$ 

In addition let us also introduce a global indexing for $\tilde{T}$, denoted by tilde values and
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corresponding to the notation for \( T \). Thus we write \( \tilde{T} = \{ \tilde{T}_j \mid j \in \tilde{I} \} \), \( \tilde{S}_{jl} \) denotes the face between \( \tilde{T}_j, \tilde{T}_l \), \( \mathbf{n}_{jl} \) the outward normal on \( \tilde{S}_{jl} \) with respect to \( \tilde{T}_j \), \( \tilde{N}(j) \) the index set of neighboring simplices to \( \tilde{T}_j \), and \( \tilde{V} := \{ \tilde{p}_k \mid k \in \tilde{K} \} \) the set of vertices of \( \tilde{T} \).

Note that the set \( \tilde{K} \) can be seen as a direct sum of \( I \) and \( K \), as we have the relation \( \tilde{V} = \{ x_j \mid j \in I \} \cup \{ p_k \mid k \in K \} \).

Having defined the dual mesh \( \tilde{T} \), we are now going to define \( c^n_h \) as a globally continuous piecewise linear function on this dual triangulation. We therefore define \( V_h \subset H^1(\mathbb{R}^d) \) as

\[
V_h := \{ w_h \in C^0(\mathbb{R}^d) \mid w_h|_T \in P_1 \text{ for all } T \in \tilde{T} \}.
\]

The piecewise linear functions \( N_k \in V_h, k \in \tilde{K} \) denote the base functions of \( V_h \) with the property \( N_k(\tilde{p}_k) = 1 \) and \( N_k(\tilde{p}_i) = 0 \) for all \( k \neq i, k, i \in \tilde{K} \).

**Definition 4.1 (Approximate solution \( u_h \))** For \( t^n \in J \) the approximate solution \( u^n_h \in V_h \) is defined through the values

- \( u_h^n(x_j) := u^n_j, \) for all \( j \in I, \) where \( u^n_j \) is defined in (6),

- \( u_h^n(p_k) := \frac{1}{|\tilde{K}|} \sum_{\{j \mid T_j \cap \Omega_k \neq \emptyset\}} u^n_j |T_j \cap \Omega_k|, \) for all \( k \in K \).

Furthermore we define the space-time function \( u_h \) by

\[
 u_h(\cdot, 0) := u^n_h, \quad u_h(\cdot, t) := u^{n+1}_h, \text{ for } t \in (t^n, t^{n+1}].
\]

**Remark 4.2** For the definition of \( u_h \), we reconstruct values of \( u_h \) at the vertices of the primal mesh from the values in the element nodes \( (x_j)_{j \in I} \). This is done in such a manner, that the following conservation property is satisfied:

\[
\int_{\Omega_k} u_h^n(x) \, dx = \sum_{\{j \mid T_j \cap \Omega_k \neq \emptyset\}} \int_{T_j \cap \Omega_k} u^n_j.
\]
Secondly we remark, that with this choice of $u_h$ we have the following important property:

$$
\frac{|S_{jl}|}{d_{jl}} (A(u_{^l}^{n+1})) - A(u_{^j}^{n+1})) = \int_{S_{jl}} \nabla I_h A(u_h^{n+1}) \cdot n_{jl},
$$

(9)

where $I_h$ denotes the Lagrange interpolation operator into $V_h$ and thus $\nabla I_h A(u_h^{n+1}) \cdot n_{jl}$ is constant along the edge $S_{jl}$.

With this definition of $u_h$ we may write the cell-centered finite volume scheme (6) equivalently as:

$$
u_{^j}^{n+1} - u_{^l}^{n} - \Delta t \frac{|T_j|}{|T_j|} \int_{S_{jl}} \nabla I_h A(u_h^{n+1}) \cdot n_{jl} = \Delta t g_{^j}^{n+1}.
$$

Lemma 4.3 (Variational formulation of the finite volume scheme) Let $\Pi_h$ denote the lumping operator defined as

$$
\Pi_h(u_h)|_{T_j} := u_{^j}^{n}, \forall T_j \in T.
$$

Then, the finite volume scheme (6) can be written in the following variational form:

$$
\langle \frac{\Pi_h(u_h^{n+1} - u_h^n)}{\Delta t}, \Pi_h(\phi_h) \rangle + \langle \nabla I_h A(u_h), \nabla \phi_h \rangle - \langle \Pi_h(g_h), \Pi_h(\phi_h) \rangle = 0 \quad \forall \phi_h \in V_h
$$

(10)

where $\langle . , . \rangle$ denotes the $L^2$ inner product.

**Proof** The lemma follows from the following local property of $u_h$ that was proved in [9].

$$
\int_{S_{jl}} \nabla I_h A(u_h) \cdot n_{jl}(\phi_l - \phi_j) = \sum_{k \in V_{jl}} \int_{\partial \tilde{T}_{jl}^k} \nabla I_h A(u_h)|_{T_{jl}} \cdot n \phi_h.
$$

(11)

Here, $n$ denotes the outward normal vector on $\partial \tilde{T}_{jl}^k$ with respect to $\tilde{T}_{jl}^k$. 
5 Error Estimates for the Model Problem

In this section we derive the error estimates for the full equations.

Theorem 5.1 Let the assumptions (H1)-(H3) be satisfied. For \( n \geq 1 \), let \( \epsilon_n = (\sum_{i=1}^{n} \varepsilon_i^n)^{\frac{2}{1+\gamma}} \), where \( \gamma \) is the Hölder exponent of \( A' \circ A^{-1} \), and \( \varepsilon_1^n, \varepsilon_2^n, \varepsilon_3^n \) are the error indicators defined below. Denote \( Q_n = \Omega \times (t^{n-1}, t^n) \), and define

\[
\Lambda_n = \max \left( 1, \int_{Q_n} \frac{1}{\nu(\epsilon_n, U_h)} \left( |\partial_t U_h| + |\nabla U_h| \right) \right),
\]

(12)

where for any \( z \in \mathbb{R} \), \( \nu(\epsilon_n, U_h) = \min\{A'(s) : |A(s) - A(z)| \leq \epsilon_n\} \). Then there exists a constant \( C \) depending only on the minimum angles of the meshes \( \mathcal{M}_n \), \( n = 1, \ldots, m \), such that the following a posteriori error estimate is valid:

\[
\|u^n - U_h^n\|_{L^1} \leq \varepsilon_0^n + \sum_{i=1}^{m} (\varepsilon_4^n + \varepsilon_5^n) + C \sum_{n=1}^{m} \Lambda_n^{1+\gamma} \left( \sum_{i=1}^{3} \varepsilon_i^n \right)^{\frac{2\gamma}{1+\gamma}},
\]

(13)

where the error indicators \( \varepsilon_i^n \), \( i = 0, \ldots, 5 \) are defined as

\[
\begin{align*}
\varepsilon_0^n &= \|u_0 - U_h^n\|_{L^1(\Omega)}, \text{ initial error} \\
\varepsilon_1^n &= \tau_n^{1/2} h_n^{1/2} \sum_{(j,l) \in \xi} |\nabla I_h A(u_h^n) \cdot n_{j,l}| s_{j,l}, \text{ jump residual} \\
\varepsilon_2^n &= \tau_n^{1/2} h_n R^n \|_{L^2(\Omega)}, \text{ interior residual} \\
\varepsilon_3^n &= \tau_n^{1/2} \sum_{j \in I} h_j \int_{T_j} |\nabla I_h A(u_h^n)| dx, \text{ interior residual} \\
\varepsilon_4^n &= \int_{t^{n-1}}^{t^n} \|U_h^n - \overline{U_h}^{n-1} - (\partial_t U_h + div f(U_h))\|_{L^1(\Omega)} dt, \text{ characteristic} \\
\varepsilon_5^n &= \int_{t^{n-1}}^{t^n} \|g - \overline{g}^n\|_{L^1(\Omega)} dt, \text{ source}
\end{align*}
\]

Proof In the proof we use the Clement interpolation operator: \( \Pi^n : H^1_0(\Omega) \rightarrow V^n_0 \), which satisfies the following local approximation properties [3], for any \( \phi \in H^1_0(\Omega) \),

\[
\|\phi - \Pi^n \phi\|_{L^2(K)} + h_K|\nabla (\phi - \Pi^n \phi)|_{L^2(K)} \leq C^* h_k \|\nabla \phi\|_{L^2(N(K))},
\]

(15)

\[
\|\phi - \Pi^n \phi\|_{L^2(\Omega)} \leq C^* h_k^{1/2} \|\nabla \phi\|_{L^2(N(\Delta))},
\]

(16)
where $N(A)$ is the union of all elements in $\mathcal{M}^n$ surrounding the sets $A = K \in \mathcal{M}^n$ or $A = e \in \mathcal{B}^n$. The constant $C^{*}$ depends only on the minimum angle of the mesh $\mathcal{M}^n$.

Further our interior residual is given as

$$R^n = \bar{g}^n - \frac{U^n_h - \bar{U}_h^{n-1}}{\tau^n} + I_h A(U^n_h),$$

where $I_h$ is the Lagrange interpolation operator, that maps into the $V^n$ space.

We have the following assumption:

$$\phi = H_e(A(u_e) - A(u))\Phi \leq 1,$$

where $\Phi$ is a special test function.

We can derive the residual as follows:

$$\langle \mathcal{R}, \phi \rangle = \langle \mathcal{R}, \phi \rangle - \langle \partial_t u_h - \Pi(u_h^{n+1}) - \Pi(u_h^n), \phi - \phi_h \rangle$$

By adding the residuum and the continuous form, we get:

$$\langle R, \phi \rangle = \langle g - g_h, \phi \rangle + \langle R^n - \Delta I_h A(U^n_h), \phi - \Pi^n \phi \rangle$$

where we have the projection and the residuum problem.
We formulate once again:

\[ \langle R, \phi \rangle = \langle \partial_t u_h - \frac{\partial \Pi(u_h)}{\partial t}, \phi \rangle \]

\[ + \langle \nabla A(u_h) - \nabla I_h A(u_h), \nabla \phi \rangle + \langle g - g_h, \phi \rangle \]

\[ + \langle \partial_t \Pi_h(u_h) + \Delta I_h A(u_h) - g_h, \phi_h - \phi \rangle \]

\[ - \sum_j \langle [\nabla I_h A(u_h) \cdot n]_{S_{ij}}, \phi_h - \phi \rangle_{S_{ij}}. \]

We make the following assumption:

\[ \phi = H_{\epsilon}(A(u_{\epsilon}) - A(u)) \Phi \leq 1, \]

where \( \Phi \) is a special test function.

\[ \| \frac{1}{h} (\phi_h - \phi) \|_{L^2(T_j)} \leq C \| \nabla \phi \|_{L^2(T_j)} \]

\[ \| \frac{1}{\sqrt{h}} (\phi_h - \phi) \|_{L^2(S_{ij})} \leq C \| \nabla \phi \|_{L^2(S_{ij})} \]

We get the reformulated result:

\[ -\langle R, \phi \rangle \leq C \left( \| \partial_t u_h - \partial_t \Pi(u_h) - g_h \|^2 \right) \]

\[ + \| \sqrt{h} [\nabla I_h A(u_h)]_{ij} \|^2 \right)^{1/2} \| \nabla \phi \|_{L^2} \]

\[ + \langle \nabla (A(u_h) - I_h A(u_h), \Phi H_{\epsilon'}(A(u_{\epsilon}) - A(u)) \nabla (A(u_h) - A(u)) \rangle \]

where we did the following modifications.

We use the Schwarz inequality and the Youngs inequality:

\[ I = \langle \nabla (A(u_h) - I_h A(u_h), \Phi H_{\epsilon'}(A(u_{\epsilon}) - A(u)) \nabla (A(u_h) - A(u)) \rangle \]

\[ \leq \| \sqrt{\Phi H_{\epsilon'}(A(u_{\epsilon}) - A(u))(\nabla (A(u_h) - I_h A(u_h)) \| \]

\[ \| \sqrt{\Phi H_{\epsilon'}(A(u_{\epsilon}) - A(u))(\nabla (A(u_h) - A(u)) \| \]

\[ \leq \frac{1}{4} \| \sqrt{\Phi H_{\epsilon'}(A(u_{\epsilon}) - A(u))(\nabla (A(u_h) - I_h A(u_h)) \| \]

\[ + C \frac{1}{\epsilon} \| \nabla (A(u_h) - A(u)) \|^2 \]
where we assume:

\[
\left( \sqrt{\Phi H'(A(u_h) - A(u))} \right) \leq \frac{1}{\varepsilon}.
\] (25)

Our error estimate is

\[
-\langle R, \phi \rangle \leq \| \partial_t u_h - \frac{\partial \Pi(u_h)}{\partial t} \| + \| g - g_h \| + C \frac{1}{\varepsilon} \| h \partial_h \Pi(u_h) - g_h \|^2 \\
+ C \frac{1}{\varepsilon} \| \sqrt{h} |\nabla I_h A(u_h)|_{ij} \|^2 \\
+ C \frac{1}{\varepsilon} \| \nabla A(u_h) - I_h A(u_h) \|^2 \\
+ \frac{1}{2} \| \sqrt{\Phi H'(A(u_h) - A(u))} \nabla (A(u_h) - A(u)) \|^2.
\] (26)

6 Conclusion

We have presented a model for the nonlinear heat transport, e.g. inside a technical apparatus for crystal growth of SiC single crystals. We introduce the model as nonlinear convection diffusion equations. The equations are discretized by the finite volume method and characteristic methods. The a posteriori error estimates are derived with the Kruzkov technique. We could proof correct convergence order error and recover a standard a posteriori error estimates known for parabolic equations in the $L^\infty(L^1)$-norm. In our future work, we concentrate on further implementations and numerical methods for a crystal growth model. We will also include probabilistic and statistical concepts for the initial conditions of the parabolic equations.

References


