Solution of eighth-order boundary-value problems using nonpolynomial spline

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Abstract

We use non-polynomial spline function to develop numerical methods for the solution of the eighth-order linear boundary-value problems. End conditions of the spline are derived. We applied the presented method to an example and compared our results with the results produced by decomposition method and polynomial spline method. However, it is observed that our approach produce better numerical solutions in the sense that maximum absolute error $|e_i|$ is a minimum.

Keywords: Eighth-order boundary-value problem, Non-polynomial spline functions, End conditions, Numerical results.

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1 Introduction

We consider the eighth-order linear boundary-value problems of the form

$$y^{(8)}(x) + g(x)y(x) = q(x), \quad x \in [a, b],$$

with boundary conditions

$$y(a) = A_0, y^{(2)}(a) = A_1, y^{(4)}(a) = A_2, y^{(6)}(a) = A_3,$$

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where $A_i, B_i$ for $i = 0, 1, 2, 3$ are finite real constants and the functions $g(x)$ and $q(x)$ are continuous on $[a, b]$.

Eighth-order differential equations govern the physics of some hydrodynamic stability problems. When an infinite horizontal layer of fluid is heated from below and is subjected to the action of rotation, instability sets in. When this instability sets in as over stability, it is modeled by an eighth-order ordinary differential equation [1-3]. Agarwal [4] presented the theorems which listed the conditions for the existence and uniqueness of solutions of eighth-order boundary-value problems. Boutayeb and Twizell [5] developed the finite difference methods for the solution of (1) with different boundary conditions. Twizell et al. [6] developed numerical methods for eighth, tenth and twelfth-order eigenvalue problems arising in thermal instability. Siddiqi and Twizell [3] presented the solution of (1) with different boundary conditions using octic spline. Inc and Evans [1] presented the solutions of eighth-order boundary-value problems using Adomian decomposition method. The eighth-order boundary-value problem using Nonic spline have been solved by Ghazala Akram, Shahid S. Siddiqi [7]. In this paper we used non-polynomial spline approximation to develop a family of new numerical methods to obtain smooth approximations to the solution of eighth-order differential equation. The spline functions proposed in this paper have the form $T_9 = \text{span}\{1, x, x^2, x^3, x^4, x^5, x^6, x^7, \cos(kx), \sin(kx)\}$ where $k$ is the frequency of the trigonometric part of the spline functions which can be real or pure imaginary and which will be used to raise the accuracy of the method. Thus in each subinterval $x_i \leq x \leq x_{i+1}$, we have

$$\text{span}\{1, x, x^2, x^3, x^4, x^5, x^6, x^7 \cos(|k|x), \sin(|k|x)\}, \text{ or}$$

$$\text{span}\{1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9\}, \text{ (when } k \rightarrow 0).$$
In this paper, in Section 2, the new non-polynomial spline methods are developed for solving equation (1) along with boundary condition (2). Development of the boundary formulas are considered in Section 3. Section 4 is devoted to numerical experiment, discussion and comparison with other known methods.

2 Numerical methods

To develop the spline approximation to the eighth-order boundary-value problem (1)-(2), the interval \([a,b] \) is divided into \(n\) equal subintervals using the grid \(x_i = a + ih, i = 0, 1, ..., n\), where \(h = \frac{b-a}{n}\). We define the following non-polynomial Nonic spline \(S_i(x)\) is each subinterval \([x_i, x_{i+1}]\), \(i = 0, 1, ..., n-1\),

\[
S_i(x) = a_i \cos k(x - x_i) + b_i \sin k(x - x_i) + c_i (x - x_i)^7 + d_i (x - x_i)^6 + e_i (x - x_i)^5
\]

\[+f_i (x - x_i)^4 + g_i^*(x - x_i)^3 + r_i (x - x_i)^2 + q_i^*(x - x_i) + p_i,\]  

(4)

where \(a_i, b_i, c_i, d_i, e_i, f_i, g_i^*, r_i, q_i^*\) and \(p_i\) are real finite constants and \(k\) is free parameter. The spline \(S\) is defined in terms of its 2nd, 4th, 6th and 8th derivatives and we denote these values at knots as:

\[
S_i(x_i) = y_i, S_i^{(2)}(x_i) = m_i, S_i^{(4)}(x_i) = M_i, S_i^{(6)}(x_i) = Z_i, S_i^{(8)}(x_i) = L_i,\
\]

\[
S_i(x_{i+1}) = y_{i+1}, S_i^{(2)}(x_{i+1}) = m_{i+1}, S_i^{(4)}(x_{i+1}) = M_{i+1}, S_i^{(6)}(x_{i+1}) = Z_{i+1}, S_i^{(8)}(x_{i+1}) = L_{i+1},
\]

for \(i = 0, 1, 2, ..., n - 1\).  

(5)

Assuming \(y(x)\) to be the exact solution of the boundary value problem (1) and \(y_i\) be an approximation to \(y(x_i)\), obtained by the spline \(S(x_i)\), we can obtained the coefficients in (3) in the following form

\[
a_i = \frac{h^8 L_i}{\theta^8}, \quad b_i = -h \frac{\coth(\theta) L_i - \csc(\theta) L_{i+1}}{\theta^8}, \quad c_i = -\frac{h^2 (L_i - L_{i+1}) + \theta^2 (Z_i - Z_{i+1})}{5040 h \theta^2},
\]
\[ e_i = -\frac{1}{720\theta^4}[6\theta^2(L_{i+1} - L_i) + \theta^4(2\theta^2 L_i + L_{i+1}) + 6\frac{\theta^6}{h}(M_i - M_{i+1}) + h\theta^4(2Z_i + Z_{i+1})], \]
\[ f_i = \frac{\theta^4 M_i - h^4 L_i}{24\theta^4}, \]
\[ g_i^* = -\frac{1}{2160}[360h^5(L_i - L_{i+1}) - 60h^5\theta^2(2L_i + L_{i+1}) - h^5\theta^4(8L_i + 7L_{i+1}) + 360\frac{\theta^6}{h}(M_i - M_{i+1}) + 60h\theta^6(2M_i + M_{i+1}) - (8h^3\theta^6 Z_i + 7h^3\theta^6 Z_{i+1})], \]
\[ r_i = \frac{\theta^6 m_i + h^6 L_i}{2\theta^6}, \]
\[ q_i^* = -\frac{1}{15120\theta^8}[15120h^7(L_{i+1} - L_i) + h^7 L_i(5040\theta^2 + 336\theta^4 + 32\theta^6) + h^7 L_{i+1}(2520\theta^2 + 294\theta^4 + 31\theta^6) + 2520h\theta^8(2m_i + m_{i+1}) - h^3\theta^8(336M_i + 294M_{i+1}) + 15120\frac{\theta^8}{h}(y_i - y_{i+1}) + 32h^2\theta^8(Z_i + 31Z_{i+1})], \]
\[ p_i = \frac{\theta^8 y_i - h^8 L_i}{\theta^8}, \]

where \( \theta = kh \) and \( i = 0, 1, 2, ..., n - 1 \). Applying the continuity conditions of the first, third, fifth and seventh derivative, at knots, i.e. \( S_i^{(\lambda)}(x_i) = S_i^{(\lambda)}(x_i) \), where \( \lambda = 1, 3, 5 \) and 7 after simplifying, we get the following spline relation in terms of eighth derivative of spline \( L_i \) and \( y_i \):

\[ y_{i-4} - 8y_{i-3} + 28y_{i-2} - 56y_{i-1} + 70y_i - 56y_{i+1} + 28y_{i+2} - 8y_{i+3} + y_{i+4} = \theta^8[\alpha(L_{i-4} + L_{i+4}) + \beta(L_{i-3} + L_{i+3}) + \gamma(L_{i-2} + L_{i+2}) + \delta(L_{i-1} + L_{i+1}) + \eta L_i], \]

\[ i = 4, 5, ..., n - 4, \quad (6) \]

where

\[ \alpha = \frac{1}{\theta^8} - \frac{1}{\theta^2 \sin(\theta)} + \frac{840}{5040\theta^5 \sin(\theta)} - \frac{42}{5040\theta^3 \sin(\theta)} + \frac{1}{5040\theta \sin(\theta)}, \]

\[ \beta = \left( \frac{1}{5040\theta^4} \right) \left( \frac{-40320}{\theta} + 10080 \coth(\theta) - 1680\theta^2 \coth(\theta) + 84\theta^4 \coth(\theta) \right) \]
\[ -2\theta^6 \coth(\theta) + \frac{30240}{\sin(\theta)} - \frac{1008\theta^4}{\sin(\theta)} + \frac{120\theta^6}{\sin(\theta)} \],

\[ \gamma = \left( \frac{1}{5040\theta^7} \right) \left( \frac{141120}{\theta} - 60480 \coth(\theta) + 2016\theta^4 - 240\theta^6 \coth(\theta) \right) \]
\[ -80640 \frac{\theta^2}{\sin(\theta)} - \frac{672\theta^4}{\sin(\theta)} - \frac{672\theta^4}{\sin(\theta)} + \frac{1192\theta^6}{\sin(\theta)} \],

\[ \delta = \left( \frac{1}{5040\theta^7} \right) \left( \frac{-282240}{\theta} + 151200 \coth(\theta) + 15120\theta^2 \coth(\theta) \right) \]
+1260\theta^4 \coth(\theta) - 2382\theta^6 \coth(\theta) + \frac{131040}{\sin(\theta)} + \frac{13440\theta^2}{\sin(\theta)} + \frac{2352\theta^4}{\sin(\theta)} + \frac{2536\theta^6}{\sin(\theta)},
\eta = \left(\frac{1}{5040\theta^7}\right)\left(\frac{35280}{\theta} - 201600 \coth(\theta) - 26880\theta^2 \coth(\theta)\right)
-672\theta^4 \coth(\theta) - 4832\theta^6 \coth(\theta) - \frac{151200}{\sin(\theta)} - \frac{15120\theta^2}{\sin(\theta)} - \frac{1260\theta^4}{\sin(\theta)} + \frac{2382\theta^6}{\sin(\theta)},

If \(\theta \to 0\) then \((\alpha, \beta, \gamma, \delta, \eta) \to \left(\frac{1}{352880}, \frac{502}{352880}, \frac{14608}{352880}, \frac{362880}{352880}, \frac{156190}{352880}\right),\) and we get polynomial nonic spline functions.

### 3 Development of the boundary formulas

To obtain unique solution we need six more equations to be associate with (5) so that we use the following boundary conditions. In order to obtain the eighth-order boundary formula we define the following identity:

\[
\sum_{k=0}^{5} a_k' y_{k+1} + b' h y_0^{(2)} + c' h^2 y_0^{(4)} + d' h^3 y_0^{(6)} + f' h^8 y_0^{(8)} + h^8 \sum_{k=0}^{7} p_k y_{k+1}^{(8)} + t_1 = 0, \quad (7)
\]

\[
\sum_{k=0}^{5} a_k'' y_{k+1} + b'' h y_0^{(2)} + c'' h^2 y_0^{(4)} + d'' h^3 y_0^{(6)} + f'' h^8 y_0^{(8)} + h^8 \sum_{k=0}^{7} p_k y_{k+1}^{(8)} + t_2 = 0, \quad (8)
\]

\[
\sum_{k=0}^{5} a_k''' y_{k+2} + b' h y_0^{(2)} + c' h^2 y_0^{(4)} + d' h^3 y_0^{(6)} + f' h^8 y_0^{(8)} + h^8 \sum_{k=0}^{7} p_k y_{k+2}^{(8)} + t_3 = 0, \quad (9)
\]

\[
\sum_{k=0}^{5} a_k^{**} y_{k+n-7} + b' h^2 y_n^{(2)} + c' h^4 y_n^{(4)} + d' h^6 y_n^{(6)} + f' h^8 y_n^{(8)} + h^8 \sum_{k=0}^{7} p_k y_{k+n-9}^{(8)} + t_{n-3} = 0, \quad (10)
\]

\[
\sum_{k=0}^{5} a_k^{***} y_{k+n-6} + b' h^2 y_n^{(2)} + c' h^4 y_n^{(4)} + d' h^6 y_n^{(6)} + f' h^8 y_n^{(8)} + h^8 \sum_{k=0}^{7} p_k y_{k+n-8}^{(8)} + t_{n-2} = 0, \quad (11)
\]

\[
\sum_{k=0}^{5} a_k^{****} y_{k+n-5} + b' h^2 y_n^{(2)} + c' h^4 y_n^{(4)} + d' h^6 y_n^{(6)} + f' h^8 y_n^{(8)} + h^8 \sum_{k=0}^{7} p_k y_{k+n-7}^{(8)} + t_{n-1} = 0, \quad (12)
\]
where all of the coefficients are arbitrary parameters to be determined. In order to obtain the eighth-order method we find that:

\[
(a_0', a_1', a_2', a_3', a_4', a_5') = \left( -\frac{30512401920}{32807189}, -\frac{91537205760}{32807189}, -\frac{104613949440}{32807189}, \right. \\
\left. \frac{58845346560}{32807189}, \frac{17435658240}{32807189}, \frac{2179457280}{32807189}, \right)
\]

\[
(b', c', d', f') = \left( -\frac{10897286400}{32807189}, -\frac{3450807360}{32807189}, \frac{1483241760}{32807189}, \frac{300986881}{164035945}, \right.
\left. \frac{223398508}{32807189}, \frac{590137111}{164035945}, \frac{1}{32807189}, \right)
\]

\[
(p_0', p_1', p_2', p_3', p_4', p_5', p_6') = \left( 1, -\frac{5701644687}{164035945}, \frac{1177531667}{164035945}, \frac{1875711107}{164035945}, \right.
\left. \frac{123398508}{32807189}, \frac{164035945}{32807189}, \frac{32807189}{32807189}, \right)
\]

\[
(a_0'', a_1'', a_2'', a_3'', a_4'', a_5'') = \left( -\frac{163459296000}{1435435681}, \frac{57210753600}{1435435681}, \frac{79005326400}{1435435681}, \right.
\left. \frac{54486432000}{1435435681}, \frac{190702512000}{1435435681}, \frac{27243216000}{1435435681}, \right)
\]

\[
(b'', c'', d'', f'') = \left( -\frac{27243216000}{1435435681}, \frac{24972948000}{1435435681}, \frac{22778355600}{1435435681}, \frac{61909835873}{1435435681}, \right.
\left. \frac{783334445}{2870871362}, \frac{2308124087}{2870871362}, \frac{1}{2870871362}, \right)
\]

\[
(p_0'', p_1'', p_2'', p_3'', p_4'', p_5'', p_6'') = \left( -1, \frac{49247037297}{2870871362}, \frac{18406646727}{2870871362}, \frac{103296178340}{2870871362}, \right.
\left. \frac{164128290155}{2870871362}, \frac{4783334445}{2870871362}, \frac{1}{2870871362}, \right)
\]

\[
(a_0^0, a_1^0, a_2^0, a_3^0, a_4^0, a_5^0) = \left( -\frac{119870150400}{43872082867}, \frac{23974030000}{43872082867}, \frac{688708500480}{43872082867}, \right.
\left. \frac{208909289920}{43872082867}, \frac{30512401920}{43872082867}, \right)
\]

\[
(b^0, c^0, d^0, f^0) = \left( -\frac{10897286400}{1482079820171}, \frac{18707008320}{1482079820171}, \frac{29452943520}{1482079820171}, \frac{75535200984}{1482079820171}, \right.
\left. \frac{119870150400}{43872082867}, \frac{30512401920}{43872082867}, \frac{688708500480}{43872082867}, \right)
\]

\[
(p_0^0, p_1^0, p_2^0, p_3^0, p_4^0, p_5^0, p_6^0) = \left( -1, \frac{5462130939737}{219360414335}, \frac{11653635480064}{219360414335}, \right.
\left. \frac{1482079820171}{219360414335}, \frac{1202116771179}{219360414335}, \frac{6067413845377}{219360414335}, \frac{174559333366}{219360414335}, \right)
\]

\[
(a_0^*, a_1^*, a_2^*, a_3^*, a_4^*, a_5^*) = \left( -\frac{30512401920}{43872082867}, \frac{198330612480}{43872082867}, \frac{520890289920}{43872082867}, \right.
\left. \frac{688708500480}{43872082867}, \frac{23974030000}{43872082867}, \frac{119870150400}{43872082867}, \right)
\]

\[
(b^*, c^*, d^*, f^*) = \left( -\frac{10897286400}{43872082867}, \frac{18707008320}{43872082867}, \frac{29452943520}{43872082867}, \frac{75535200984}{43872082867}, \right.
\left. \frac{119870150400}{43872082867}, \frac{30512401920}{43872082867}, \frac{688708500480}{43872082867}, \right)
\]
At the mesh point \( x_i \) the proposed differential equation (1) may be discretized by:

\[
L_i = q_i - g_i y_i,
\]
where \( L_i = S_i^{(8)}(x_i), q_i = q(x_i), g_i = g(x_i) \) and \( y_i = y(x_i) \).

The local truncation error corresponding to the method (5) is given by

\[
t_i = (1 - (2\alpha + 2\beta + 2\gamma + 2\delta + \eta))h^8 y_i^{(8)} + \left( \frac{1}{3} - (16\alpha + 9\beta + 4\gamma + \delta) \right)h^{10} y_i^{(10)}
\]

\[
+ \left( \frac{19}{360} - \frac{1}{12} (256\alpha + 81\beta + 16\gamma + \delta) \right)h^{12} y_i^{(12)} + \left( \frac{1}{189} - \frac{1}{360} (4096\alpha + 729\beta + 64\gamma + \delta) \right)h^{14} y_i^{(14)}
\]

\[
+ \left( \frac{457}{1209600} - \frac{1}{20160} (65536\alpha + 6561\beta + 256\gamma + \delta) \right)h^{16} y_i^{(16)}
\]

\[
+ \left( \frac{491}{23950080} - \frac{1}{1814400} (1048576\alpha + 59049\beta + 1024\gamma + \delta) \right)h^{18} y_i^{(18)}
\]

\[
+ O(h^{19}), \ i = 4, 4, ..., n - 4.
\]

By using the above truncation error to eliminate the coefficients of various powers \( h \) we can obtain classes of the methods. For any choice of \( \alpha, \beta, \gamma, \delta \) and \( \eta \) whose \( 2\alpha + 2\beta + 2\gamma + 2\delta + \eta = 1 \), We obtain the class of methods.

**Remark(i): Second-order method**

For \( (\alpha, \beta, \gamma, \delta) = (\frac{1}{362880}, \frac{502}{362880}, \frac{1468}{362880}, \frac{88234}{362880}) \) and \( \eta = \frac{156190}{362880} \), we obtain the second-order method with truncation error \( t_i = -\frac{17138293}{205757496}h^{10} y_i^{(10)} + O(h^{11}) \).

**Remark(ii): Fourth-order method**

For \( (\alpha, \beta, \gamma) = (0, 0, 0) \) and \( \delta = \frac{1}{3} \), we obtain the fourth-order method with truncation error \( t_i = \frac{1}{40}h^{12} y_i^{(12)} + O(h^{13}) \).

**Remark(iii): Sixth-order method**

For \( (\alpha, \beta, \gamma) = (0, 0, \frac{1}{40}) \) and \( \delta = \frac{7}{20} \), we obtain the sixth-order method with truncation error \( t_i = \frac{1}{5040}h^{14} y_i^{(14)} + O(h^{15}) \).

**Remark(iv): Eight-order method**

For \( (\alpha, \beta, \gamma) = (0, \frac{1}{5040}, \frac{1}{12}) \) and \( \delta = \frac{397}{1680} \), we obtain the eight-order method with truncation error \( t_i = -\frac{1}{1209600}h^{16} y_i^{(16)} + O(h^{17}) \).

**Remark(v): Tenth-order method**

For \( (\alpha, \beta, \gamma) = (\frac{-1}{1209600}, \frac{31}{151200}, \frac{7193}{302400}) \) and \( \delta = \frac{35737}{151200} \), we obtain the tenth-order method.
with truncation error \( t_i = \frac{1}{1330560} h^{18} y_i^{(18)} + O(h^{19}) \).

The spline solution of boundary value problem (1) is determined, using boundary formulas (7) – (12) and the main formula (6). This yields the system of \((N - 1)(N - 1)\) linear equations. Considering \( Y = [y_1, y_2, ..., y_{n-1}]^T \) \( C = [c_1, c_2, ..., c_{n-1}]^T \), \( y_i \) satisfies the following system in matrix form as: \((A + h^8 BG)Y = C\), where

\[
A = \begin{pmatrix}
    a_1' & a_2' & a_3' & a_4' & a_5' & 0 & 0 \\
    a_0'' & a_1'' & a_2'' & a_3'' & a_4'' & a_5'' & 0 & 0 \\
    0 & a_0^\circ & a_1^\circ & a_2^\circ & a_3^\circ & a_4^\circ & a_5^\circ & 0 & 0 \\
    -8 & 28 & -56 & 70 & -56 & 28 & -8 & 1 & 0 \\
    1 & -8 & 28 & -56 & 70 & -56 & 28 & -8 & 1 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    1 & -8 & 28 & -56 & 70 & -56 & 28 & -8 & 1 \\
    0 & a_0^\circ & a_1^\circ & a_2^\circ & a_3^\circ & a_4^\circ & a_5^\circ & 0 & 0 \\
    0 & a_0^\circ \star & a_1^\circ \star & a_2^\circ \star & a_3^\circ \star & a_4^\circ \star & a_5^\circ \star & 0 & 0 \\
    0 & a_0^\circ \star & a_1^\circ \star & a_2^\circ \star & a_3^\circ \star & a_4^\circ \star & a_5^\circ \star & 0 & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\tag{17}
\]

and

\[
B = \begin{pmatrix}
p_1' & p_2' & p_3' & p_4' & p_5' & p_6' & p_7' \\
p_0'' & p_1'' & p_2'' & p_3'' & p_4'' & p_5'' & p_6'' & p_7'' \\
0 & p_0^\circ & p_1^\circ & p_2^\circ & p_3^\circ & p_4^\circ & p_5^\circ & p_6^\circ & p_7^\circ \\
-\beta & -\gamma & -\delta & -\eta & -\delta & -\gamma & -\beta & -\alpha \\
-\alpha & -\beta & -\gamma & -\delta & -\eta & -\delta & -\gamma & -\beta & -\alpha \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-\alpha & -\beta & -\gamma & -\delta & -\eta & -\delta & -\gamma & -\beta & -\alpha \end{pmatrix}
\tag{18}
\]
$G = diag(g_i), i = 1, 2, 3, ..., n - 1.$

The vector $C$ is defined by

$$c_i = \left\{ \begin{array}{ll}
-d'_0 y_0 - b' h^2 y_0^{(2)} - c' h^4 y_0^{(4)} - d' h^6 y_0^{(6)} - f' h^8 y_0^{(8)} \\
-h^8 (p'_0 g_0 + p'_1 q_1 + p'_2 q_2 + p'_3 q_3 + p'_4 q_4 + p'_5 q_5) \\
+p'_6 q_6 + p'_7 q_7, & i = 1, \\
-b'' h^2 y_0^{(2)} - c'' h^4 y_0^{(4)} - d'' h^6 y_0^{(6)} - f'' h^8 y_0^{(8)} - h^8 (p''_0 q_1 + p''_1 q_2 + p''_2 q_3) \\
+p''_3 q_4 + p''_4 q_5 + p''_5 q_6 + p''_6 q_7 + p''_7 q_8), & i = 2, \\
-b^* h^2 y_0^{(2)} - c^* h^4 y_0^{(4)} - d^* h^6 y_0^{(6)} - f^* h^8 y_0^{(8)} - h^8 (p^*_0 q_2 + p^*_1 q_3 + p^*_2 q_4 + p^*_3 q_5) \\
+p^*_4 q_6 + p^*_5 q_7 + p^*_6 q_8 + p^*_7 q_9), & i = 3, \\
-(1 + \alpha h^8 g_0) y_0 + h^8 [\alpha (q_0 + q_8) + \beta (q_1 + q_7) + \gamma (q_2 + q_6) + \delta (q_3 + q_5)] \\
+\eta (q_4), & i = 4, \\
-h^8 [\alpha (q_{i-4} + q_{i+4}) + \beta (q_{i-3} + q_{i+3}) + \gamma (q_{i-2} + q_{i+2}) + \delta (q_{i-1} + q_{i+1})] \\
+\eta (q_i), & i = 5, ..., n - 5, \\
-(1 + \alpha h^8 g_n) y_0 + h^8 [\alpha (q_n + q_{n-8}) + \beta (q_{n-1} + q_{n-7}) + \gamma (q_{n-2} + q_{n-6})] \\
+\delta (q_{n-3} + q_{n-5}) + \eta (q_{n-4}), & i = n - 4, \\
-b^* h^2 y_n^{(2)} - c^* h^4 y_n^{(4)} - d^* h^6 y_n^{(6)} - f^* h^8 y_n^{(8)} - h^8 (p^*_0 q_{n-2} + p^*_1 q_{n-3} + p^*_2 q_{n-4}) \\
+p^*_3 q_{n-5} + p^*_4 q_{n-6} + p^*_5 q_{n-7} + p^*_1 q_{n-8} + p^*_6 q_{n-9}), & i = n - 3, \\
-b^{**} h^2 y_n^{(2)} - c^{**} h^4 y_n^{(4)} - d^{**} h^6 y_n^{(6)} - f^{**} h^8 y_n^{(8)} - h^8 (p^{**}_0 q_{n-8} + p^{**}_1 q_{n-7}) \\
+p^{**}_2 q_{n-6} + p^{**}_3 q_{n-5} + p^{**}_1 q_{n-4} + p^{**}_5 q_{n-3} + p^{**}_0 q_{n-2} + p^{**}_7 q_{n-1}), & i = n - 2, \\
-a^{*}_5 y_n - b^* h^2 y_n^{(2)} - c^* h^4 y_n^{(4)} - d^* h^6 y_n^{(6)} - f^* h^8 y_n^{(8)} \\
-h^8 (p^*_1 (q_{n-y} + q_n) + p^*_0 q_{n-1} + p^*_2 q_{n-2} + p^*_3 q_{n-3} + p^*_4 q_{n-4} + p^*_5 q_{n-5}) \\
+p^*_1 q_{n-6} + p^*_0 q_{n-7}), & i = n - 1, \\
\end{array} \right. \right.$$
4 Numerical results

Example 1. We consider the following boundary-value problem:

\[ y^{(8)}(x) + xy = -(48 + 15x + x^3)e^x, \quad 0 \leq x \leq 1, \]

\[ y(0) = 0, y^{(2)}(0) = 0, y^{(4)}(0) = -8, y^{(6)}(0) = -24, \]

\[ y(1) = 0, y^{(2)}(1) = -4e, y^{(4)}(1) = -16e, y^{(6)}(1) = -36e. \]

The exact solution for this problem is \( y(x) = x(1 - x)e^x. \)

This problem has been solved by the authors [2,3]. We applied our methods to solve this problem with different value of \( h \) and parameters \( \alpha, \beta, \gamma \) and \( \delta \). The computed solutions are compared with the exact solution at grid points. The observed maximum absolute errors are tabulated in Table 1 and compared with the methods in [2,3]. It has been observed that our methods are more efficient.

5 Conclusion

We approximate solution of the eighth-order linear boundary-value problems by using non-polynomial spline. The new methods enable us to approximate the solution at every point of the range of integration. Table 1 shows that our methods produced better numerical results in the sense that \( \max|e_i| \) is minimum in comparison with the methods in [2,3].
Table 1: Observed maximum absolute errors for example (1)

<table>
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<th>x</th>
<th>$\alpha = \frac{1}{1209600}$, $\beta = \frac{502}{502880}$</th>
<th>$\alpha = 0, \beta = 0$</th>
<th>$\alpha = 0, \beta = 0$</th>
<th>$\alpha = 0, \beta = \frac{1}{5040}$, $\gamma = \frac{913}{22680}$, $\delta = \frac{44117}{181440}$</th>
<th>$\gamma = 0, \delta = \frac{1}{3}$</th>
<th>$\gamma = \frac{1}{42}, \delta = \frac{7}{30}$</th>
<th>$\gamma = \frac{1}{42}, \delta = \frac{397}{10580}$</th>
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</table>


<table>
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References


