A two-parameter family of second-order iterative methods for solving non-linear equations

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Abstract

In this paper, we suggest and analyze a family of iterative methods for solving non-linear equations \(f(x) = 0\), that it can derive the Newton’s method. It is shown that the order of convergence of this methods is two or higher. Several numerical examples are given to illustrate the performance of the presented methods.

Keywords: Newton’s method, Iterative methods, Non-linear equations, Order of convergence, Chebyshev’s method.

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1 Introduction

In this paper, we consider a family of iterative methods to find a simple root \(\alpha\), i.e., \(f(\alpha) = 0\) and \(f'(\alpha) \neq 0\), of a non-linear equation \(f(x) = 0\). Probably the most well-known and widely used algorithm to find a root of single non-linear equation is Newton’s method. This method is written as

\[x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.\] (1)

This is an important and basic method, which converges quadratically. We know that iterative methods for finding the approximate solutions of the non-linear equations are being developed using several different techniques including Taylor series, quadrature

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formulas, Homotopy and decomposition techniques.

A family of third-order methods, called Chebyshev-Halley methods [2], is defined by

\[ x_{n+1} = x_n - \left(1 + \frac{1}{2} \frac{l_f(x_n)}{1 - \alpha l_f(x_n)} \right) \frac{f(x_n)}{f'(x_n)}, \]  

(2)

where

\[ l_f(x_n) = \frac{f''(x_n)f(x_n)}{f'^2(x_n)}. \]  

(3)

In this paper, by using Chebyshev-Halley methods, we introduce new iterative methods.

The rest of this paper is organized as follows. In section 2, we give the method based on Chebyshev’s method. In section 3, we illustrate the result with some numerical examples and in the last section, the conclusions are presented.

2 The methods and analysis of convergence

We consider the Chebyshev-Halley methods given by

\[ x_{n+1} = x_n - \left(1 + \frac{l_f(x_n)}{2} \right) \frac{f(x_n)}{f'(x_n)}, \]  

(4)

where

\[ l_f(x_n) = \frac{f''(x_n)f(x_n)}{f'^2(x_n)}. \]  

(5)

Our aim is to find a correction term for the method defined by (4)-(5) that will yield a second-order method. To do this, first consider fitting the function \( f(x) \) around the point \((x_n, f(x_n))\) with the third-degree polynomial

\[ g(x) = ax^3 + bx^2 + cx + d. \]  

(6)

Imposing the tangency condition at the \( n \)th iterate \( x_n \)

\[ g'(x_n) = f'(x_n), \]  

(7)

according to (6), we have

\[ c = f'(x_n) - 3ax_n^2 - 2bx_n, \]  

(8)
thereby obtaining the first derivative of the approximating polynomial as
\[ g'(x) = 3ax^2 + 2bx + f'(x_n) - 3ax_n^2 - 2bx_n. \] (9)

By \( y_n = x_n - \frac{f(x_n)}{f'(x_n)} \), we approximate \( f'(y_n) \) as
\[ f'(y_n) \approx g'(y_n) = \frac{f^2(x_n) + 3af^2(x_n) + (-6ax_n - 2b)f(x_n)}{f'(x_n)}. \] (10)

So, we consider \( f''(x_n) \approx \frac{f'(y_n) - f'(x_n)}{y_n - x_n} \), and obtain
\[ f''(x_n) \approx \left(f'(x_n) - f'(y_n)\right)\frac{f'(x_n)}{f(x_n)}, \] (11)

by substitution of above approximating in (5), we have
\[ l_f(x_n) \approx \frac{f'(x_n) - f'(y_n)}{f'(x_n)}, \] (12)

so from (10) and (12), we obtain
\[ l_f(x_n) \approx \frac{(2\lambda x_n - \mu)f(x_n) - \lambda f^2(x_n)}{f^2(x_n)}. \] (13)

Now by (13), we get the new two-parameter family of methods defined by
\[ x_{n+1} = x_n - (1 + (2\lambda x_n - \mu)f(x_n) - \lambda f^2(x_n))\frac{f(x_n)}{2f^2(x_n)}, \] (14)

where \( \lambda = 3a, \mu = -2b \). For the method defined by (14), we have

**Theorem.** Let \( \alpha \in I \) be a simple root of a sufficiently differentiable function \( f : I \to \mathbb{R} \) for an open interval \( I \), then the methods defined by (14) have a minimum order of convergence equal to two and it satisfies the following error equation:
\[ e_{n+1} = (c_2 - \frac{2\lambda \alpha - \mu}{2f'(\alpha)}) e_n^2 + \cdots, \] (15)

where \( c_2 = \frac{f''(\alpha)}{2f'(\alpha)} \) and \( \lambda, \mu \in R \).

**Proof:** Let \( \alpha \) be a simple zero of \( f \), \( e_n = x_n - \alpha \) and \( h_n = z_n - \alpha \). By using Taylor expansion around \( x = \alpha \) and taking into account \( f(\alpha) = 0 \), we have
\[ f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + \cdots], \] (16)
\[ f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + \cdots], \]  
\[ f(z_n) = f'(\alpha)[h_n + c_2h_n^2 + c_3h_n^3 + \cdots], \]  
\[ f'(z_n) = f'(\alpha)[1 + 2c_2h_n + 3c_3h_n^2 + 4c_4h_n^3 + \cdots], \]

where \( c_k = \frac{f^{(k)}(\alpha)}{k!f'(\alpha)}. \) We know

\[ (1 + a)^{-1} = \sum_{k \geq 0} \left( \frac{-1}{k} \right) a^k, \]  

then

\[ \frac{1}{f'(x_n)} = \frac{1}{f'(\alpha)}[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + \cdots]^{-1} \]
\[ = \frac{1}{f'(\alpha)} \sum_{k \geq 0} \left( \frac{-1}{k} \right) (2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + \cdots)^k. \]  

For simplicity, we compute

\[ (2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + \cdots)^k, \] for \( k=2,3. \)

\[ (2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + \cdots)^2 = 4c_2^2e_n^2 + 12c_2c_3e_n^3 + \cdots, \]

\[ (2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + \cdots)^3 = 8c_2^3e_n^3 + \cdots. \]

By substitution of above expansions in (21), we have

\[ \frac{1}{f'(x_n)} = \frac{1}{f'(\alpha)}[1 - 2c_2e_n + (-3c_3 + 4c_2^2)e_n^2 + (-4c_4 + 12c_2c_3 - 8c_2^3)e_n^3 + \cdots], \]  

and hence from (16) and (22), we have

\[ \frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + (2c_2^2 - 2c_3)e_n^3 + \cdots. \]  

By (16), we get

\[ f^2(x_n) = f^2(\alpha)[e_n^2 + 2c_2e_n^3 + \cdots]. \]
Since $e_n = x_n - \alpha$, we compute

$$
(2\lambda x_n - \mu) = 2\lambda \ (e_n + \alpha) - \mu = (2\lambda \ \alpha - \mu) + 2 \ \lambda \ e_n,
$$

so from (16) and (25), we obtain

$$
(2\lambda x_n - \mu) f(x_n) = f'(\alpha)[(2\lambda \ \alpha - \mu) e_n + (2\lambda \ \alpha c_2 - \mu \ c_2 + 2\lambda) e_n^2 +
(2\lambda \ \alpha c_3 - \mu \ c_3 + 2 \ \lambda \ c_2) e_n^3 + \cdots].
$$

By (17), we have

$$
f'^2(x_n) = f'^2(\alpha)[1 + 4c_2 e_n + (6c_3 + 4c_2^2)e_n^2 + (8c_4 + 12c_2 c_3)e_n^3 + \cdots],
$$

according to (20) and (27), we get

$$
\frac{1}{f'^2(x_n)} = \frac{1}{f'^2(\alpha)} \sum_{k \geq 0} \binom{-1}{k} [4c_2 e_n + (6c_3 + 4c_2^2)e_n^2 + (8c_4 + 12c_2 c_3)e_n^3 + \cdots]^k,
$$

where

$$
[4c_2 e_n + (6c_3 + 4c_2^2)e_n^2 + (8c_4 + 12c_2 c_3)e_n^3 + \cdots]^2 = 16c_2^2 e_n^2 + 8c_2(6c_3 + 4c_2^2)e_n^3 + \cdots,
$$

$$
[4c_2 e_n + (6c_3 + 4c_2^2)e_n^2 + (8c_4 + 12c_2 c_3)e_n^3 + \cdots]^3 = 64c_2^3 e_n^3 + \cdots,
$$

therefore

$$
\frac{1}{f'^2(x_n)} = \frac{1}{f'^2(\alpha)}[1 - 4c_2 e_n + (-6c_3 + 12c_2^2)e_n^2 + (-8c_4 + 36c_2 c_3 - 32c_2^3)e_n^3 + \cdots].
$$

Now by (24) and (26), we have

$$
(2\lambda x_n - \mu) f(x_n) - \lambda \ f^2(x_n) = f'^2(\alpha)[A e_n + B e_n^2 + C e_n^3 + \cdots],
$$

where $A = \frac{2\lambda \ \alpha - \mu}{f'(\alpha)}$ and $B = \frac{2\lambda \ \alpha c_2 - \mu \ c_2 + 2\lambda}{f'(\alpha)} - \lambda$ and

$$
C = -2c_2 \lambda + \frac{2\lambda \ \alpha c_3 - \mu \ c_3 + 2\lambda \ c_2}{f'(\alpha)}.
$$
So
\[
1 + \frac{(2\lambda x_n - \mu)f(x_n) - \lambda f^2(x_n)}{2f'^2(x_n)} = 1 + \frac{A}{2}e_n + \frac{B - 4Ac_2}{2}e_n^2 + \frac{(C - 4c_2B - 6Ac_3 + 12Ac_2^2)}{2}e_n^3 + \cdots,
\]
by (23) and (31), we get
\[
(1 + \frac{(2\lambda x_n - \mu)f(x_n) - \lambda f^2(x_n)}{2f'^2(x_n)} \frac{f(x_n)}{f'(x_n)} = e_n + (-c_2 + \frac{A}{2})e_n^2 +
(2c_2^2 - 2c_3 - \frac{Ac_2}{2} + \frac{B - 4Ac_2}{2})e_n^3 + \cdots.
\]
Now according to (14) and (32), we have
\[
e_{n+1} = e_n - (1 + \frac{(2\lambda x_n - \mu)f(x_n) - \lambda f^2(x_n)}{2f'^2(x_n)} \frac{f(x_n)}{f'(x_n)} = (c_2 - \frac{A}{2})e_n^2 + \cdots,
\]
or
\[
e_{n+1} = (c_2 - \frac{A}{2})e_n^2 + \cdots,
\]
this means that the proof is completed, i.e.,
\[
e_{n+1} = (c_2 - \frac{2\lambda \alpha - \mu}{2f'(\alpha)})e_n^2 + \cdots.
\]

3 Numerical Examples

In this section, all computations were done using MATHEMATICA using 120 digit floating point arithmetic (Digits:=120). We accept an approximate solution rather than the exact root, depending on the precision ($\epsilon$) of the computer. We use the following stopping criteria for computer programs:

\[
(i)|x_{n+1} - x_n| < \epsilon, \quad (ii)|f(x_{n+1})| < \epsilon,
\]
and so, when the stopping criterion is satisfied, $x_{n+1}$ is taken as the exact root $\alpha$ computed. For numerical illustrations, we used the fixed stopping criterion $\epsilon = 10^{-15}$. 
We present some numerical test results to illustrate the efficiency of the new iterative methods in Table 1. We compare the Newton’s method (NM), the Sharma method [3] (SM), defined by

\[ x_{n+1} = x_n + \frac{2f(x_n)}{f'(x_n) + Sgn(f'(x_n))\sqrt{f'^2(x_n) - 4af(x_n)}}, \quad n \geq 0 \] (33)

the Changbum Chun method [1] (CM), defined by

\[ x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{f'^2(x_n) - af(x_n)}, \] (34)

with \( a = 1 \) and (EAM) method defined by (14) with \( \lambda = 2 \) and \( \mu = 1 \), introduced in the present contribution. We used the following functions.

\[ f_1(x) = x^3 + 4x^2 - 10, \]
\[ f_2(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5, \]
\[ f_3(x) = \sin(x)e^x + \ln(x^2 + 1), \]
\[ f_4(x) = \sin^2(x) - x^2 + 1, \]
\[ f_5(x) = x + \sin(x) - 2, \]
\[ f_6(x) = (x - 1)^3 - 2, \]
\[ f_7(x) = e^{x^2+7x-30} - 1. \]

4 Conclusions

In this paper, we defined and analyzed a family of iterative methods for solving nonlinear equations and proved that the order of convergence of these methods is at least two.
Table 1. Comparison of the number of iterations in (NM), (SM) and (EAM) methods.

<table>
<thead>
<tr>
<th>$f(x)$, $x_0$</th>
<th>NM</th>
<th>SM</th>
<th>CM</th>
<th>EAM</th>
</tr>
</thead>
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<td>$f_1(x)$, $x_0 = 1$</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>$f_2(x)$, $x_0 = −1$</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$f_3(x)$, $x_0 = 5$</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>$f_4(x)$, $x_0 = 2$</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>$f_5(x)$, $x_0 = 0$</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>$f_6(x)$, $x_0 = 4$</td>
<td>7</td>
<td>7</td>
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<td>7</td>
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<tr>
<td>$f_7(x)$, $x_0 = 4$</td>
<td>19</td>
<td>19</td>
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References

