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On Spaces of Fourier-Stieltjes Transform of Vector Measures on Compact Groups

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Abstract

In [1], the author extended to vector measures on a compact non commutative group the notion of Fourier-Stieltjes transform. This brings to light some functional spaces. In this paper, we study some of their topological properties. In particular, we found their dual spaces.

Keywords: Fourier transform, Compact group, Banach space

1 Introduction

The Fourier transform of a complex valued function on a commutative locally compact group G, such as $\mathbb{R}^n$, is again a complex valued function on the character group X of G. Otherwise, it is a familly $(E_\sigma)_{\sigma\in\Sigma}$ of continuous linear operators $E_\sigma : H_\sigma \to H_\sigma$, where Σ is the dual object of the compact non commutative group G, and σ, a class of irreducible unitary representations of G in a Hilbert space H_σ.

In case $\mathbb{C}$ is replaced by a Banach space A, it is a familly of continuous sesquilinear mappings $\phi(\sigma) : H_\sigma \times H_\sigma \to A$. In fact, for each $\sigma \in \Sigma$, we choose once and

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for all an element U^σ in σ, denote its representation space by H_σ, and fix an orthonormal basis $(\xi_1^\sigma, \ldots, \xi_{d_\sigma}^\sigma)$ of H_σ, where $d_\sigma = dim H_\sigma$, as a canonical basis. We put $u_{ij}^\sigma(t) = \langle U_t^\sigma \xi_j^\sigma, \xi_i^\sigma \rangle$ and introduce the operator $\overline{U}^\sigma$ on H_σ such that $\langle \overline{U}_t^\sigma \xi_j^\sigma, \xi_i^\sigma \rangle = \overline{u_{ij}^\sigma(t)}$, the complex conjugate of $u_{ij}^\sigma(t)$ The Fourier-Stieltjes transform on G for an A-valued bounded vector measure m, where A is a normed space, is given by :

$$\hat{m}(\sigma)(\xi, \eta) = \int_G \langle \overline{U}_t^\sigma \xi, \eta \rangle \, dm(t) \quad (\xi, \eta) \in H_\sigma \times H_\sigma.$$

(For details on vector measures See [4] and [5]). The mapping $H_\sigma \times H_\sigma \to A, (\xi, \eta) \to \hat{m}(\sigma)(\xi, \eta)$ is a continuous and sesquilinear [1].

This generates a certain number of interesting linear spaces $\mathcal{S}_p(\Sigma, A)$ that we specify as follow.

We write $\prod\limits_{\sigma \in \Sigma} \mathcal{S}(H_\sigma \times H_\sigma, A) = \mathcal{S}(\Sigma, A)$ where $\mathcal{S}(H_\sigma \times H_\sigma, A)$ is the space of continuous sesquilinear mappings from $H_\sigma \times H_\sigma$ into A. $\mathcal{S}(\Sigma, A)$ is a linear space with addition and multiplication by scalars, defined coordinatewise. For $\phi \in \mathcal{S}(\Sigma, A)$, we put :

$$\|\phi\|_\infty = \sup\{\|\phi(\sigma)\| | \sigma \in \Sigma\},$$

with $\|\phi(\sigma)\| = \sup\{\|\phi(\sigma)(\xi, \eta)\| \mid \|\xi\| \leq 1, \|\eta\| \leq 1\}$. We denote by

$\mathcal{S}_\infty(\Sigma, A)$, the space $\{\phi \in \mathcal{S}(\Sigma, A) | \|\phi\|_\infty < \infty\}$,

$\mathcal{S}_{00}(\Sigma, A)$, the space $\{\phi \in \mathcal{S}_\infty(\Sigma, A) | \{\sigma \in \Sigma | \phi(\sigma) \neq 0\}$ is finite $\}$

and $\mathcal{S}_0(\Sigma, A)$, the space

$\{\phi \in \mathcal{S}_\infty(\Sigma, A) | \forall \varepsilon > 0, \{\sigma \in \Sigma | \|\phi(\sigma)\| > \varepsilon\}$is finite $\}$.

In [2] it is proved that :

1. The mapping $\phi \to \|\phi\|_\infty$ is a norm on $\mathcal{S}_\infty(\Sigma, A)$, and $\mathcal{S}_\infty(\Sigma, A)$ is a Banach space with respect to this norm.

2. $\mathcal{S}_{00}(\Sigma, A)$ is dense in $\mathcal{S}_0(\Sigma, A)$.

3. Every $\phi(\sigma) \in \mathcal{S}(H_\sigma \times H_\sigma, A)$ is determined by the d_σ^2 elements $a_{ij}^\sigma = \phi(\sigma)(\xi_j^\sigma, \xi_i^\sigma)$ of A. More precisely, we have :
   $\phi(\sigma) = \sum\limits_{i,j=1}^{d_\sigma} d_\sigma a_{ij}^\sigma \hat{u}_{ij}^\sigma(\sigma)$, $\hat{u}_{ij}^\sigma$ being the Fourier transform of u_{ij}^σ.

4. The mapping $m \to \hat{m}$ from $M^1(G, A)$, the space of A-valued bounded measures on G into $\mathcal{S}_\infty(\Sigma, A)$, is linear, injective and continuous.

2 Main Results

2.1 The spaces $\mathcal{S}_p(\Sigma, A)$ $1 \le p \le \infty$

We define :

$$\mathcal{S}_p(\Sigma, A) = \{\phi \in \mathcal{S}(\Sigma, A) | \sum_\sigma d_\sigma \sum_{ij} \|\phi(\sigma)(\xi_j^\sigma, \xi_i^\sigma)\|^p < \infty\}, \quad 1 \le p < \infty,$$

and $\mathcal{S}_\infty(\Sigma, A)$ as in the introduction. They are linear spaces for pointwise operations. We define a norm on $\mathcal{S}_p(\Sigma, A)$ by

$$\|\phi\|_p = \left(\sum_{\sigma \in \Sigma} d_\sigma \sum_{i,j} \|\phi(\sigma)(\xi_j^\sigma, \xi_i^\sigma)\| \right)^{\frac{1}{p}}.$$

Theorem 2.1 *For each p, $1 \le p \le \infty$, the space $\mathcal{S}_p(\Sigma, A)$ is a Banach space.*

Proof. The case $p = \infty$ was done in [2] .

Let (ϕ_n) be a Cauchy sequence from the space $\mathcal{S}_p(\Sigma, A)$. Then for each $\sigma \in \Sigma$, the sequence $(\phi_n(\sigma))_n$ is a Cauchy sequence from the space $\mathcal{S}(H_\sigma \times H_\sigma, A)$ which is known to be a Banach space. Thus there exists $\phi(\sigma) \in \mathcal{S}(H_\sigma \times H_\sigma, A)$ such that

$$\lim_{n \to \infty} \|\phi_n(\sigma) - \phi(\sigma)\| = 0. \tag{1}$$

Set $\alpha_{ij}^\sigma = \phi(\sigma)(\xi_j^\sigma, \xi_i^\sigma)$ and for all n, $a_{ij}^{\sigma,n} = \phi_n(\sigma)(\xi_j^\sigma, \xi_i^\sigma)$.

We consider $\varepsilon > 0$. Since (ϕ_n) is a Cauchy sequence, then there exists $n_0 \in \mathbb{N}$ such that

$$\forall r, s \ge n_0,\ \|\phi_r - \phi_s\|_p < \varepsilon^{\frac{1}{p}} \tag{2}$$

$$i.e \quad \sum_\sigma d_\sigma \sum_{i,j} \|a_{ij}^{\sigma,r} - a_{ij}^{\sigma,s}\|^p < \varepsilon. \tag{3}$$

Letting s tends to infinity in (3), we have :

$$\sum_{\sigma} d_\sigma \sum_{i,j} \|a_{ij}^{\sigma,r} - \alpha_{ij}^{\sigma}\|^p < \varepsilon \tag{4}$$

$$i.e \quad \|\phi_r - \phi\|_p < \varepsilon \text{ pour } r \geq n_0. \tag{5}$$

$$\begin{aligned} \text{We have } \|\phi\|_p &= \|\phi - \phi_r + \phi_r\|_p \\ &\leq \|\phi - \phi_r\|_p + \|\phi_r\|_p \\ &\leq \varepsilon + \|\phi_r\|_p < \infty. \end{aligned}$$

Hence $\phi \in \mathcal{S}_p(\Sigma, A)$. Finally (5) shows that (ϕ_n) converges to ϕ in $\mathcal{S}_p(\Sigma, A)$. $\square$

2.2 Duality in the spaces $\mathcal{S}_p(\Sigma, A)$

Theorem 2.2 *Let p, q be such that $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and A^* be the dual of A. Then the space $(\mathcal{S}_p(\Sigma, A))^*$ is isometric to $\mathcal{S}_q(\Sigma, A^*)$.*

Proof. The proof of the case $p = 1$ (which implies $q = \infty$) can be found in [9]. Now, let $1 < p < \infty$. Let $T : \mathcal{S}_q(\Sigma, A^*) \to (\mathcal{S}_p(\Sigma, A))^*, \varphi \mapsto T\varphi$ be defined by $\langle T\varphi, \psi\rangle = \sum\limits_{\sigma\in\Sigma} d_\sigma \sum\limits_{i,j} \langle b_{ij}^\sigma, a_{ij}^\sigma\rangle$, $\psi \in \mathcal{S}_p(\Sigma, A)$ where $b_{ij}^\sigma = \varphi(\sigma)(\xi_j^\sigma, \xi_i^\sigma)$ and $a_{ij}^\sigma = \psi(\sigma)(\xi_j^\sigma, \xi_i^\sigma)$. Then the Theorem is a consequence of the following three lemmas.

Lemma 2.3 *The mapping T is linear and bounded.*

Proof. The linearity of T is trivial. Let us show that it is bounded.

$$\begin{aligned}
\text{We have } |\langle T\varphi,\psi\rangle| &= |\sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j}\langle b_{ij}^\sigma, a_{ij}^\sigma\rangle| \\
&\leq \sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j} |\langle b_{ij}^\sigma, a_{ij}^\sigma\rangle| \\
&\leq \sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j} \|b_{ij}^\sigma\|\|a_{ij}^\sigma\| \\
&\leq \sum_{\sigma\in\Sigma}\sum_{i,j} d_\sigma^{\frac{1}{q}}\|b_{ij}^\sigma\| d_\sigma^{\frac{1}{p}}\|a_{ij}^\sigma\| \\
&\leq \left(\sum_{\sigma\in\Sigma}\sum_{i,j} d_\sigma\|b_{ij}^\sigma\|^q\right)^{\frac{1}{q}}\left(\sum_{\sigma\in\Sigma}\sum_{i,j} d_\sigma\|a_{ij}^\sigma\|^p\right)^{\frac{1}{p}} \\
&\leq \|\varphi\|_q\|\psi\|_p.
\end{aligned}$$

So $\|T\varphi\| \leq \|\varphi\|_q$ and therefore T is bounded with $\|T\| \leq 1.\square$

Lemma 2.4 *The equality* $\|T\| = 1$ *holds.*

Proof. From part 1, we have $\|T\| \leq 1$. Let us show now that $\|T\| \geq 1$.

Take $a \in A$, such that $\|a\| = 1$. Since $a \neq 0$, we know from Functional analysis that there exists $b^* \in A^*$ such that $\|b^*\| = 1$ and $\langle b^*, a\rangle = \|a\| = 1$.

Given a fixed $\tau \in \Sigma$, we use the Kronecker symbol δ_{ij} to define $\psi_\tau \in \mathcal{S}_p(\Sigma, A)$ by

$$\psi_\tau(\sigma)(\xi_j^\sigma, \xi_i^\sigma) = a_{ij}^\sigma = \begin{cases} d_\tau^{-\frac{2}{p}} a\delta_{ij} \text{ if } \sigma = \tau \\ \\ 0 \text{ if } \sigma \neq \tau \end{cases}$$

and φ_τ in $\mathcal{S}_q(\Sigma, A^*)$ by :

$$\varphi_\tau(\sigma)(\xi_j^\sigma, \xi_i^\sigma) = b_{ij}^\sigma = \begin{cases} d_\tau^{-\frac{2}{q}} b^*\delta_{ij} \text{ if } \sigma = \tau \\ \\ 0 \text{ if } \sigma \neq \tau. \end{cases}$$

$$\text{We have } \|\varphi_\tau\|_q^q = \sum_{\sigma\in\Sigma} d_\sigma \sum_{ij} \|b_{ij}^\sigma\|^q = \sum_{\sigma\in\Sigma} d_\sigma \sum_{ij} \|d_\tau^{-\frac{2}{q}} b^*\delta_{ij}\|^q = d_\tau d_\tau d_\tau^{-2} = 1$$

$$\text{and } \|\psi_\tau\|_p^p = \sum_\sigma d_\sigma \sum_{ij} \|a_{ij}^\sigma\|^p = \sum_\sigma d_\sigma \sum_{ij} \|d_\tau^{-\frac{2}{p}} a\delta_{ij}\|^p = 1.$$

$$\begin{aligned}
\text{As such, } \langle T\varphi_\tau, \psi_\tau\rangle &= \sum_\sigma d_\sigma \sum_{ij} \langle b_{ij}^\sigma, a_{ij}^\sigma\rangle \\
&= \sum_\sigma d_\sigma \sum_{ij} \langle d_\tau^{-\frac{2}{q}} b^*\delta_{ij}, d_\tau^{-\frac{2}{p}} a\delta_{ij}\rangle \\
&= d_\tau \sum_i \langle d_\tau^{-\frac{2}{q}} b^*, d_\tau^{-\frac{2}{p}} a\rangle \\
&= d_\tau^2 \left(d_\tau^{\frac{1}{q}+\frac{1}{p}}\right)^{-2} \langle b^*, a\rangle = 1 = \|\varphi\|_q \|\psi\|_p.
\end{aligned}$$

Hence $\|T\| \geq 1$. Finally $\|T\| = 1$. □

Lemma 2.5 *The mapping T is surjective.*

Proof. In fact let $f \in (\mathcal{S}_p(\Sigma, A))^*$. For $\tau \in \Sigma$, let

$$V_\tau = \{\psi \in \mathcal{S}_p(\Sigma, A) \,|\, \psi(\sigma) = 0 \text{ if } \sigma \neq \tau, \sigma \in \Sigma\}.$$

For $\psi \in V_\tau$, let $a_{ij}^\tau = \psi(\tau)(\xi_j^\tau, \xi_i^\tau)$, $i, j = 1, \ldots, d_\tau$. There exists linear forms $b_{ij}^\tau \in A^*, i, j = 1, \ldots, d_\tau$ such that $\langle f, \psi\rangle = d_\tau \sum\limits_{i,j} \langle b_{ij}^\tau, a_{ij}^\tau\rangle$. In fact, given d_τ^2 scalars λ_{ij}^τ such that $\sum\limits_{ij} \lambda_{ij}^\tau = \frac{\langle f,\psi\rangle}{d_\tau}$, there exists $b_{ij} \in A^*$ with $\langle b_{ij}, a_{ij}^\tau\rangle = 1$; denoting $b_{ij}^\tau = \lambda_{ij}^\tau b_{ij}$, we have what is required.

Now, let us consider an element ϕ of $\mathcal{S}_{00}(\Sigma, A)$.

Since $\mathcal{S}_{00}(\Sigma, A)$ is a subset of $\mathcal{S}_2(\Sigma, A)$, one can write, according to the Riesz-Fischer theorem, $\phi = \sum\limits_{\tau\in\Sigma} d_\tau \sum\limits_{ij} a_{ij}^\tau \hat{u}_{ij}^\tau$. In fact, there exists a finite subset Σ' of Σ such that $\phi = \sum\limits_{\tau\in\Sigma'} d_\tau \sum\limits_{ij} a_{ij}^\tau \hat{u}_{ij}^\tau$. Putting $\phi_\tau = d_\tau \sum\limits_{ij} a_{ij}^\tau \hat{u}_{ij}^\tau$,
we have $\phi = \sum\limits_{\tau\in\Sigma'} \phi_\tau$. It is clear that ϕ_τ belongs to V_τ because, for $\sigma \neq \tau$, $\hat{u}_{ij}^\tau(\sigma) = 0$ (Schur's orthogonality property), so $\phi_\tau(\sigma) = \sum\limits_{ij} a_{ij}^\tau \hat{u}_{ij}^\tau(\sigma) = 0$. Thus, there exist linear forms $b_{ij}^\tau \in A^*, i, j = 1, \ldots, d_\tau$ such that

$$\langle f, \phi_\tau\rangle = d_\tau \sum_{i,j} \langle b_{ij}^\tau, a_{ij}^\tau\rangle.$$

Now by linearity of f,

$$\langle f, \phi \rangle = \sum_{\tau \in \Sigma'} d_\tau \sum_{i,j} \langle b_{ij}^\tau, a_{ij}^\tau \rangle.$$

Defining φ by :

$\varphi(\tau)(\xi_j^\tau, \xi_i^\tau) = b_{ij}^\tau$ if $\tau \in \Sigma'$ and $\varphi(\tau)(\xi_j^\tau, \xi_i^\tau) = 0$ otherwise,

we have $\varphi \in \mathcal{S}_{00}(\Sigma, A^*) \subset \mathcal{S}_q(\Sigma, A^*)$ and $\langle f, \phi \rangle = \langle T\varphi, \phi \rangle$. This means that the continuous linear forms f and $T\varphi$ coïncide on $\mathcal{S}_{00}(\Sigma, A)$ which is a dense subset of $\mathcal{S}_p(\Sigma, A)$.

Hence $f = T\varphi$. $\square$

The three lemmas show that T is an isometry from $\mathcal{S}_q(\Sigma, A^*)$ onto $(\mathcal{S}_p(\Sigma, A))^*$.

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