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On Associated and Supported Primes

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Abstract

Let R be a commutative ring with identity and M be a unitary R-module. The notions of associated and supported prime submodules of a module are defined. Our aim is to extend the results concerning associated and supported prime ideals of a module to associated and supported prime submodules of a module and find new properties when the modules are multiplication, have finite length, are finitely generated or weakly finitely generated.

Keywords: Associated prime submodule, Multiplication module, Weakly finitely generated module, Prime submodule, Support of a module, Radical of a submodule.

1 Introduction

This is the first of three papers, which will be written in the sequel, concerning associated, supported prime ideals and associated, supported prime submodules of a module. We believe an author gains no score by making the proof of a result artificially sophisticated and referring the reader to wonder among various papers. Therefore our aim is to give self-contained proofs, as much as possible, using the basic definitions and notions and if a known result is proved, our proof is a new one. We hope these three papers throw a good deal of light on the subject and make further study and research on associated and supported prime submodules more convenient.

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In this paper all the rings are commutative with identity and all the modules are unitary. If K and N are submodules of an R- module M we define $(N :_R K) = (N : K) = \{r \in R | rK \subseteq N\}$, which is an ideal of R. A proper submodule N of an R-module M is said to be prime if for every $r \in R$ and $x \in M$; $rx \in N$ implies that $x \in N$ or $r \in (N : M)$. In this case clearly $P = (N : M)$ is a prime ideal of R and we say that N is P-prime. We recall that an R-module M is called a multiplication module if for any submodule N of M there exists an ideal I of such that $N = IM$. It can be shown that M is a multiplication module if and only if $N = (N : M)M$, for any submodule N of M. If S is a non-empty subset of an R-module M, the annihilator of S, denoted by $Ann_R(S)$ or simply $Ann(S)$, is defined as $\{r \in R | rS = 0\}$. The sets of all prime ideals of R and all prime submodules of an R-module M are denoted by $Spec(R)$ and $Spec(M)$, respectively.

A proper submodule N of an R-module M is called semiprime if for every submodule K of M and ideal I of R, $I^2K \subseteq N$ implies that $IK \subseteq N$. Equivalently, N is semiprime if for every $x \in M$, $r \in R$, $m \in \mathbb{Z}^+$ from $r^m x \in N$ we find that $rx \in N$. For the proof see [8, Example 2.4 page 753]. It is clear that every prime submodule of an R-module is semiprime. Also since the ring R is an R-module by itself, we can similarly define a semiprime ideal of a ring. These are needed when we use [4, Corollary 3.9]

In Section 2 we define the associated and supported prime ideals and prove some properties related to them. This section prepares a ground for parts of the results stated in Section 3. Our main work begins in Section 3 when we define associated and supported prime submodules of a module. Here some results of Section 2 are extended to associated and supported prime submodules. Also a number of statements are proved when the module is multiplication, weakly finitely generated or has a finite length.

2 Associated and Supported Prime Ideals

In this section we concentrate mostly on associated and supported prime ideals of a module in order to have a background and better understanding of associated and supported prime submodules of a module, which will be considered afterwards.

Definition 2.1. Let M be an R-module and P be a prime ideal of R. The ideal P is called an *associated prime ideal* of M if P is the annihilator of some non-zero $x \in M$, that is, for some $x \in M$, $x \neq 0$, $P = Ann_R(x) = \{a \in R | ax = 0\} = (0 : x)$. The set of all associated prime ideals of M is denoted by $Ass_R(M)$.

Lemma 2.2. *Let M be an R-module. Then the following statements are equivalent:*
(i)$M = 0$;
(ii)$M_P = 0$ *for all* $P \in Spec(R)$;
(iii) $M_m = 0$ *for all maximal ideals* m *of* R.

Proof. See [7, Lemma 9.15].

Definition 2.3. Let M be an R-module. The *support* of M, denoted by $Supp_R(M)$, is defined to be the set $\{P \in Spec(R) | M_P \neq 0\}$.

From the above lemma and definition it is clear that $M_P = 0$ for all $P \in Spec(R)$ if and only if $Supp_R(M) = \emptyset$. In the following proposition we give an alternative definition for $Supp_R(M)$.

Proposition 2.4. *Let M be an R-module. Consider the set*

$$T = \{P \in Spec(R) | P \supseteq (0 : x) = Ann_R(x) \quad for\ some \quad x \in M, x \neq 0\}.$$

Then $T = Supp_R(M)$.

Proof. Let $P \in Supp_R(M)$, then $M_P \neq 0$. Let $\overline{x} \in M_P$ be a non-zero element. Then $\overline{x} = x/s$ for some $0 \neq x \in M$ and $s \in R \setminus P$. Consider the ideal $(0 : x) = Ann_R(x)$. Let $\alpha \in Ann_R(x)$, that is, $\alpha x = 0$. If $\alpha \notin P$ then $\overline{x} = x/s = \alpha x/\alpha s = 0/s = 0_{M_P}$, a contradiction. Hence $\alpha \in P$ and we have $(0 : x) = Ann_R(x) \subseteq P$. Therefore $P \in T$. On the other hand, let $P \in T$ so $P \supseteq (0 : x) = Ann_R(x)$ for some $x \in M$, $x \neq 0$.

We localize M at P and show that $M_P \neq 0$. If $M_P = 0$ then $\overline{x} = x/s = 0_{M_P}$ for every $s \in R \setminus P$. Hence there exists $\gamma \in R \setminus P$ such that $\gamma x = 0$. But this implies $\gamma \in Ann_R(x) \subseteq P$, a contradiction. So $M_P \neq 0$, that is, $P \in Supp_R(M)$. We finally have $T = Supp_R(M)$.

The following lemma gives a better description for $Supp_R(M)$, when M is finitely generated.

Lemma 2.5. *Let M be a finitely generated R-module. Then*

$$Supp_R(M) = \{P \in Spec(R) \mid P \supseteq Ann_R(M)\}.$$

Proof. Let $P \in Supp_R(M)$ then $P \supseteq (0 : x) = Ann_R(x)$ for some non-zero element $x \in M$. Since it is clear that $Ann_R(M) \subseteq Ann_R(x)$ we have $P \supseteq Ann_R(M)$. On the other hand, let $P \in Spec(R)$ be such that $P \supseteq Ann_R(M)$. Since M is finitely generated we can write $M = Rx_1 + Rx_2 + ... + Rx_n$ for some $x_i \in M$, $1 \leq i \leq n$. But then

$$Ann_R(M) = (0 : M) = (0 : \sum_{i=1}^{n} Rx_i) = \bigcap_{i=1}^{n}(0 : Rx_i) = \bigcap_{i=1}^{n}(0 : x_i) \subseteq P.$$

Since P is a prime ideal of R there exists $1 \leq j \leq n$ such that $(0 : x_j) \subseteq P$. Therefore $P \in Supp_R(M)$ and the lemma is proved.

For the rest of this section we state and prove some important properties of $Ass_R(M)$ and $Supp_R(M)$.

Proposition 2.6. *If R is a Noetherian ring, then the R-module M is zero if and only if $Ass_R(M)$ is empty.*

Proof. By the definition of associated prime ideals of an R-module M;

$$Ass_R(M) = \{P \in Spec(R) | P = (0 : x),\ for\ some\ 0 \neq x \in M\}.$$

If $M = 0$ then it is clear that $Ass_R(M) = \emptyset$. Now, let $M \neq 0$ and consider the set

$$S = \{Ann_R(x) = (0 : x) | x \in M, x \neq 0\}.$$

Then $S \neq \emptyset$ and since R is Noetherian the set S has a maximal element. Assume that $P_0 = Ann_R(x_0) = (0 : x_0)$ is a maximal element of S, where $0 \neq x_0 \in M$. We claim that P_0 is a prime ideal of R. Let $\alpha, \beta \in R$ and $\alpha.\beta \in P_0$. Assume that $\alpha \notin P_0$ and $\beta \notin P_0$. Hence $\alpha x_0 \neq 0$ and $\beta x_0 \neq 0$. But $(\alpha\beta)x_0 = \beta(\alpha x_0) = \alpha(\beta x_0) = 0$. So $\alpha \in Ann_R(\beta x_0)$ and $\beta \in Ann_R(\alpha x_0)$. Since αx_0 and βx_0 are both non-zero elements of M, we have $Ann_R(\alpha x_0), Ann_R(\beta x_0) \in S$. It is clear that $Ann_R(x_0)$ is contained in both $Ann_R(\alpha x_0)$ and $Ann_R(\beta x_0)$. Therefore by maximality of $Ann_R(x_0)$ in S we have $P_0 = Ann_R(x_0) = Ann_R(\alpha x_0) = Ann_R(\beta x_0)$. Hence $\alpha, \beta \in Ann_R(x_0)$, a contradiction. We conclude that P_0 is a prime ideal of R and so $Ass_R(M) \neq \emptyset$.

Lemma 2.7. *Let the R-module M be Noetherian, then $Ass_R(M)$ is a finite set.*

Proof. Let $0 = \bigcap_{i=1}^{n} N_i$ be a normal decomposition for the zero submodule of M, where N_i is $P_i = \sqrt{(N_i : M)}$-primary. By [6, Theorem 13 page 105], this decomposition is unique. Let $P \in Ass_R(M)$ then $P = (0 : x) = Ann_R(x)$ for some $x \in M$, $x \neq 0$. Assume $x \notin N_i$ for $1 \leq i \leq j$ and $x \in N_i$ for $j + 1 \leq i \leq n$. By [6, Proposition 5 page 193], there exists $n_i \in Z^+$ such that $P_i^{n_i} M \subseteq N_i$. So $P_i^{n_i} x \subseteq N_i$ and hence $(\bigcap_{i=1}^{j} P_i^{n_i})x \subseteq \bigcap_{i=1}^{j} N_i$. Also $x \in \bigcap_{i=j+1}^{n} N_i$ and we have $(\bigcap_{i=1}^{j} P_i^{n_i})x \subseteq \bigcap_{i=1}^{n} N_i = 0$. Therefore $(\bigcap_{i=1}^{j} P_i^{n_i}) \subseteq Ann_R(x) = P$. Since P is prime, $P_i \subseteq P$ for some $1 \leq i \leq j$. We show that $P_i = P$. Let $\alpha \in P$ then $\alpha x = 0 \in N_i$. But $x \notin N_i$ and N_i is P_i-primary, thus $\alpha \in P_i$. Therefore any $P \in Ass_R(M)$ is one of the $P_i s$, and hence $Ass_R(M)$ is finite.

Theorem 2.8. *Let R be a Noetherian ring and S a non-empty multiplicatively closed subset of R. Then for any R-module M*

$$Ass_{S^{-1}R}(S^{-1}M) = \{S^{-1}P | P \in Ass_R(M) \text{ and } P \cap S = \emptyset\}.$$

Proof. We use [2, Proposition 3.14 and Corollary 3.15]. Let $P \in Ass_R(M)$ and $P \bigcap S = \emptyset$. Then $P = Ann_R(x)$ for some $x \in M$, $x \neq 0$. Hence

$$S^{-1}P = S^{-1}(Ann_R(x)) = S^{-1}(Ann_R(Rx)) = Ann_{S^{-1}R}(S^{-1}Rx)$$

and it is clear that $Ann_{S^{-1}R}(S^{-1}Rx) = Ann_{S^{-1}R}(x/s)$, where $s \in S$ (here $x/s \neq 0$, since $P \bigcap S = \emptyset$). Therefore $S^{-1}P = Ann_{S^{-1}R}(x/s) \in Ass_{S^{-1}R}(S^{-1}M)$. Conversely, let $P' \in Ass_{S^{-1}R}(S^{-1}M)$, then $P' = Ann_{S^{-1}R}(\overline{x})$ for some $\overline{x} \in S^{-1}M$, $\overline{x} \neq 0$. By [6, page 158 Theorem 9], there is a prime ideal P of R such that $S^{-1}P = P'$ $(P = (P')^C)$, the contraction of P' in R. We have

$$P' = Ann_{S^{-1}R}(S^{-1}Rx) = S^{-1}(Ann_R(Rx)) = S^{-1}(Ann_R(x))$$

and also $P' = S^{-1}P$. Hence $S^{-1}(Ann_R(x)) = S^{-1}P$ and we conclude that $(Ann_R(x))^S = P^S$, the S-components of $Ann_R(x)$ and P, respectively. But $P^S = P$ and hence $(Ann_R(x))^S = P$. Finally we show that $(Ann_R(x))^S = Ann_R(sx)$ for some $s \in S$. Since R is Noetherian, $(Ann_R(x))^S$ is finitely generated, $(Ann_R(x))^S = (a_1, ..., a_n)$-say. Now $a_i \in (Ann_R(x))^S = P$ implies that there exists $s_i \in S$ such that $s_i a_i \in Ann_R(x)$, that is, $s_i a_i x = 0$, $1 \leq i \leq n$. Put $s = s_1 ... s_n$, then $s a_i x = 0$ and for every $a \in (Ann_R(x))^S$, $sax = a(sx) = 0$, which implies $a \in Ann_R(sx)$. So $(Ann_R(x))^S \subseteq Ann_R(sx)$ and since it is clear that $Ann_R(sx) \subseteq (Ann_R(x))^S$, we have $(Ann_R(x))^S = Ann_R(sx)$. Therefore $P = Ann_R(sx)$ and $sx \neq 0$ (because if $sx = 0$ then $P = Ann_R(sx) = R$, a contradiction. Therefore $P \in Ass_R(M)$ and the proof is complete.

Now we prove a very useful result.

Proposition 2.9. *Let M be an R-module and $P \in Spec(R)$, where R is a Noetherian ring. Then $P \in Supp_R(M)$ if and only if $P \supseteq Q$ for some $Q \in Ass_R(M)$.*

Proof. Let $P \supseteq Q$ for some $Q \in Ass_R(M)$. Then $Q = Ann_R(x) = (0 : x)$, where $x \in M$, $x \neq 0$. By Proposition 2.4, we have $P \in Supp_R(M)$. On the other hand, let $P \in Supp_R(M)$. Since R is Noetherian the ring $S^{-1}R = R_P$ (where $S = R \backslash P$) is also Noetherian. Now $M_P \neq 0$ and so by Proposition 2.6, $Ass_{R_P}(M_P) \neq \emptyset$. By Theorem 2.8, $Ass_{R_P}(M_P) = \{S^{-1}Q | Q \in Ass_R(M)$ and $Q \cap S = \emptyset\}$. Since $S = R \backslash P$, for any Q in the above set we have $Q \cap S = \emptyset$ and hence $Q \subseteq P$. The existence of some $Q \in Ass_R(M)$ is clear by the fact that $Ass_{R_P}(M_P) \neq \emptyset$. The proof is now complete.

Lemma 2.10. *Let R be a Noetherain ring and M an R-module. Then the sets of* *minimal elements of $Ass_R(M)$ and that of $Supp_R(M)$ are equal.*

Proof. It is clear that $Ass_R(M) \subseteq Supp_R(M)$. If $P_0 \in Ass_R(M)$ is minimal in $Supp_R(M)$, then P_0 is minimal in $Ass_R(M)$. Because if $P \in Ass_R(M)$ and $P \subset P_0$, since $P_0 \in Supp_R(M)$, this contradicts the minimality of P_0 in $Supp_R(M)$. Let $P \in Ass_R(M)$ be minimal in $Ass_R(M)$. If there exists $Q_0 \in Supp_R(M)$ such that $Q_0 \subset P$, then by Proposition 2.9 there exists $P_0 \in Ass_R(M)$ such that $P_0 \subseteq Q_0$. But then $P_0 \subset P$, a contradiction to minimality of P in $Ass_R(M)$. Therefore P is minimal in $Supp_R(M)$. Finally, by using Proposition 2.9, we can show that no element of $Supp_R(M) \backslash Ass_R(M)$ can be minimal in $Supp_R(M)$.

Proposition 2.11. *Let M be an R-module. Let $Z(M)$ be the set of zero-divisors on M, that is, the set of all $r \in R$ such that $rx = 0$ for some non-zero $x \in M$. Then $\bigcup\{P|P \in Ass_R(M)\} \subseteq Z(M)$, with equality if R is a Noetherian ring.*

Proof. Let $a \in \bigcup\{P|P \in Ass_R(M)\}$ then $a \in Q$ for some $Q \in Ass_R(M)$. But $Q = Ann_R(x)$ for some non-zero $x \in M$ and so $ax = 0$. This implies $a \in Z(M)$ and therefore $\bigcup\{P|P \in Ass_R(M)\} \subseteq Z(M)$. Next let R be Noetherian and $b \in Z(M)$. Then $bx = 0$ for some non-zero $x \in M$. Consider the set $S = \{Ann_R(x)|b \in Ann_R(x)$ and $0 \neq x \in M\}$. Then $S \neq \emptyset$ and since R is Noetherian, S has a maximal element $P_0 = Ann_R(x_0)$-say. We show that P_0 is a prime ideal of R. Let $ab \in P_0$, where $a, b \in R$ and let $a \notin P_0$. Then $ax_0 \neq 0$. Now $(ab)x_0 = b(ax_0) = 0$ and so $b \in Ann_R(ax_0)$. Hence $Ann_R(ax_0) \in S$. But $Ann_R(x_0) \subseteq Ann_R(ax_0)$ and by maximality of P_0 we have $P_0 = Ann_R(x_0) = Ann_R(ax_0)$. Therefore $b \in P_0$ and P_0 is prime. So $P_0 \in Ass_R(M)$ and $b \in P_0 \subseteq \bigcup\{P|P \in Ass_R(M)\}$.

Proposition 2.12. *Let M be a module over a Noetherian ring R and let $a \in R$. Then a does not belong to any associated prime ideal of M if and only if $ax \neq 0$ for every $x \in M$, $x \neq 0$.*

Proof. Let $a \in P = Ann_R(x) \in Ass_R(M)$ for some $x \in M$, $x \neq 0$. Then it is clear that $ax = 0$. On the other hand, let $ax = 0$ for some $x \in M$, $x \neq 0$, that is, $a \in Ann_R(x)$. Consider the set $S = \{Ann_R(y)|ay = 0,\ 0 \neq y \in M\}$. Then $S \neq \emptyset$ and has a maximal element $P_0 = Ann_R(x_0)$, which is a prime ideal of R. Therefore $a \in P_0 \in Ass_R(M)$.

Lemma 2.13. *The following statements are equivalent for a proper submodule N of a multiplication R-module M:*

(i) N is prime.

(ii) $Ann_R(M/N)$ is a prime ideal.

(iii) $N = PM$ for some prime ideal P of R with $Ann_R(M) \subseteq P$.

Proof.(i) $\Rightarrow$ (ii). We know that $Ann_R(M/N) = (N : M)$. Assume $ab \in (N : M)$, where $a, b \in R$ and let $b \notin (N : M)$, that is, $bM \nsubseteq N$. So there exists $m_0 \in M$ such that $bm_0 \notin N$. Since $abM \subseteq N$ we have $(ab)m_0 = a(bm_0) \in N$. Hence $a \in (N : M)$ and $(N : M) = Ann_R(M/N)$ is a prime ideal of R.

(ii) $\Rightarrow$ (iii). Let $P = Ann_R(M/N) = (N : M)$ be a prime ideal of R. Since M is a multiplication module we have $N = (N : M)M = PM$. Also $Ann_R(M) = (0 : M) \subseteq (N : M) = P$.

(iii) $\Rightarrow$ (i). Consider the ring $T = R/Ann_R(M)$, then M is a T-module, as well as an R-module, by the rule $(r + Ann_R(M))x = rx$, for every $r \in R$, $x \in M$ and T-submodules of M are the same as R-submodules of M. It is clear that M is a faithful T-module. Now $\overline{P} = P/Ann_R(M)$ is a prime ideal of the ring T. By [4, Lemma 2.10 page 764], if $\overline{a} \in T$ and $x \in M$ satisfy $\overline{a}x \in \overline{P}M$ then $\overline{a} \in \overline{P}$ or $x \in \overline{P}M$. It is clear that $N = PM = \overline{P}M$ and if $ax \in PM$, that is, $\overline{a}x \in \overline{P}M$ then $\overline{a} \in \overline{P}$ or $x \in \overline{P}M$, that is, $a \in P$ or $x \in PM$. But $P \subseteq (PM : M)$, so $x \in PM$ or $a \in (PM : M)$. Therefore $N = PM$ is a prime submodule of M.

3 Associated and Supported Prime Submodules

In this section we define associated and supported prime submodules of a module and prove a number of results concerning them. Also we try to extend most of the results proved in Section 2, for associated and supported prime submodules.

Let P be a prime ideal of R and M be an R-module. We define $M(P) = \{x \in M | sx \in PM \text{ for some } s \in R\backslash P\}$. It is clear that $M(P)$ is a submodule of M and $PM \subseteq M(P)$.

Proposition 3.1. *Let M be an R-module and P be a prime ideal of R.*

(1) $M(P) = M$ or $M(P)$ is a P-prime submodule of M.

(2) If N is any P-prime submodule of M, then $M(P) \subseteq N$.

Proof. (1) Let $M(P) \neq M$. Assume that $rx \in M(P)$, where $r \in R$ and $x \in M$. Let $x \notin M(P)$, so for all $s \in R\backslash P$, $sx \notin PM$. Since $rx \in M(P)$ there exists $s_0 \in R\backslash P$ such that $s_0(rx) = (s_0r)x \in PM$. This implies $r \notin R\backslash P$. So $r \in P$ and since $P \subseteq (PM : M) \subseteq (M(P) : M)$ we have $r \in (M(P) : M)$. Therefore $M(P)$ is a prime submodule of M. Since $M(P) \neq M$ there exists $x_0 \in M\backslash M(P)$. Let $r \in (M(P) : M)$, that is, $rM \subseteq M(P)$ and so $rx_0 \in M(P)$. Hence there exists $s' \in R\backslash P$ such that $s'(rx_0) = (s'r)x_0 \in PM$. But $x_0 \notin PM$ and hence $s'r \in P$. We find that $r \in P$ and therefore $M(P)$ is P-prime.

(2) Let $x \in M(P)$. So there exists $s \in R\backslash P$ such that $sx \in PM$. Since N is P-prime, $P = (N : M)$, that is, $PM \subseteq N$. Hence $sx \in N$ and since $s \notin P = (N : M)$ we have $x \in N$. Hence $M(P) \subseteq N$.

Remark. Result (2) of the above proposition implies that if for $P \in Spec(R)$ there exists a P-prime submodule N, then clearly $M(P) \neq M$.

Let M be an R-module and N a submodule of M. We recall that the $M-radical$ of N, denoted by $rad_M N$ or simply $radN$, is defined as $radN = \bigcap P(P$ is a prime submodule of M and $P \supseteq N)$. If no prime submodule of M contains N, we write $radN = M$. The radical of the module M is defined to be $rad\ 0$.

Corollary 3.2.*If M is an R-module, then $rad\ 0 = \bigcap_{P \in Spec(R)} M(P)$.*

Proof. We know that $rad\ 0 = \bigcap_{K \in Spec(M)} K$. By the above proposition it is clear that $rad\ 0 \subseteq \bigcap_{P \in Spec(R)} M(P)$. Next let N be a P-prime submodule of M. If we construct

$M(P)$ then, again by the above proposition, $M(P) \subseteq N$. Hence

$$\bigcap_{P \in Spec(R)} M(P) \subseteq \bigcap_{K \in Spec(M)} K = rad\ 0.$$

Proposition 3.3. *Let M be a finitely generated R-module and let I be a radical ideal of R. Then $(IM : M) = I$ if and only if $Ann_R(M) \subseteq I$.*

Proof. See [5, Proposition 8 page 65].

Lemma 3.4. *Let M be an R-module and P be a prime ideal of R.*

(1) If $Ann_R(M) \nsubseteq P$ then $M(P) = M$.

(2) If M is finitely generated and $P \supseteq Ann_R(M)$, then $M(P)$ is a P-prime submodule of M.

Proof. (1) Let $Ann_R(M) \nsubseteq P$, so that exists $r_0 \in Ann_R(M) \backslash P$ and we have $r_0 M = 0$. Let $x \in M$ then $r_0 x = 0 \in PM$ and since $r_0 \in R \backslash P$ we have $x \in M(P)$. Hence $M(P) = M$.

(2) Let $M = Rx_1 + Rx_2 + ... + Rx_n$. Let $r \in (M(P) : M)$, that is, $rM \subseteq M(P)$. So $rx_i \in M(P)$ which implies $s_i(rx_i) \in PM$ for some $s_i \in R \backslash P$ and every $1 \leq i \leq n$. Take $s = s_1 ... s_n \in R \backslash P$ to see that $r(sx_i) \in PM \quad (1 \leq i \leq n)$. If $m \in M$ then $m = a_1x_1 + ... + a_nx_n$ for some $a_i \in R$. But then $(rs)(a_ix_i) \in PM, \quad 1 \leq i \leq n$, and hence $(rs)m \in PM$. So $(rs)M \subseteq PM$, that is, $rs \in (PM : M)$. Since the prime ideal P is radical, by Proposition 3.3, we have $(PM : M) = P$. Therefore $rs \in P$ and so $r \in P$. We see that $(M(P) : M) = P$ and this implies $M(P) \notin M$. Now by Proposition 3.1(1), $M(P)$ is a P-prime submodule of M.

Corollary 3.5. *If M is finitely generated as an R-module, then $rad\ 0 = \bigcap M(P)$,where $P \in Spec(R)$ and $P \supseteq Ann_R(M)$.*

Proof. By Lemma 3.4 (1), if $P \in Spec(R)$ and $Ann_R(M) \nsubseteq P$, then $M(P) = M$. So for this type of prime ideals P, $M(P)$ has no effect on the intersection and can be ignored. Now use Corollary 3.2.

Theorem 3.6. *Let M be an R-module and let P be a prime ideal of R. If M is finitely* *generated or multiplication, then $M(P)$ is a P-prime submodule of M if and only if $P \in Supp_R(M)$.*

Proof. Let M be a finitely generated R-module. Let $M(P)$ be a P-prime submodule of M. Then clearly $M(P) \neq M$. Assume that $m_0 \in M$ and $m_0 \notin M(P)$. If $x \in Ann_R(M)$ then $xM = 0$ and so $xm_0 = 0 \in M(P)$. Hence there exists $s \in R\backslash P$ such that $(sx)m_0 \in PM$. But $sx \notin R\backslash P$ and so $sx \in P$, which implies $x \in P$. Therefore $Ann_R(M) \subseteq P$ and by Lemma 2.5, $P \in Supp_R(M)$. On the other hand, let $P \in Supp_R(M)$ then by Lemma 2.5, $P \supseteq Ann_R(M)$ and so by Lemma 3.4(2),$M(P)$ is a P-prime submodule of M.

Next we assume that M is a multiplication R-module. Let $M(P)$ be a P-prime submodule of M, so $(M(P) : M) = P$. Let $x \in M\backslash M(P)$ and put $Ann_R(x) = A$. If $r \in Ann_R(x)$ then $rx = 0 \in M(P)$ and since $x \notin M(P)$, we have $r \in (M(P) : M) = P$. So $Ann_R(x) \subseteq P$ and by Proposition 2.4, $P \in Supp_R(M)$.

Let $P \in Supp_R(M)$, then $M_P \neq 0$. We prove that $M \neq M(P)$. Assume the contrary and suppose that $M = M(P)$. By localizing M at P we have $M_P = M(P)_P$. Now we show that $M(P)_P = (PR_P)M_P$:

$$M(P) = \{x \in M | sx \in PM \text{ for some } s \in R\backslash P\}$$

and $M(P)_P = \{\frac{x}{\delta} | x \in M(P),\ \delta \in R\backslash P\} = \{\frac{x}{\delta} | sx \in PM \text{ and } s, \delta \in R\backslash P\}$.

But $\frac{x}{\delta} = \frac{sx}{s\delta} \in (PM)_P = P_P M_P = (PR_P)M_P$. Therefore $M_P = (M(P))_P = (PR_P)M_P$. Since PR_P is the unique maximal ideal of the local ring R_P, by [3, Proposition 1] we have $M_P = 0$. This is a contradiction and so $M(P) \neq M$. The result follows by Proposition 3.1(1).

Lemma 3.7. *Let $\{M_i\}_{i \in I}$ be a family of R-modules. Then for each $P \in Spec(R)$,*

$$(\bigoplus_{i \in I} M_i)(P) = \bigoplus_{i \in I} M_i(P).$$

Proof. We have

$$(\bigoplus_{i\in I} M_i)(P) = \{x \in \bigoplus_{i\in I} M_i | sx \in P(\bigoplus_{i\in I} M_i) \text{ for some } s \in R\backslash P\}.$$

Let $x \in (\bigoplus_{i\in I} M_i)(P)$, where $x = \{x_i\}_{i\in I}$, then $sx \in P(\bigoplus_{i\in I} M_i)$ for some $s \in R\backslash P$. Hence $sx = \sum_{f.s.} a\{y_i\}_{i\in I}$, where $a \in P$, $y_i \in M_i$ and so $sx = \{sx_i\}_{i\in I} = \{\sum_{f.s.} ay_i\}_{i\in I}$. This implies $sx_i = \sum_{f.s.} ay_i \in PM_i$ and thus $x_i \in M_i(P)$. Therefore $x = \{x_i\}_{i\in I} \in \bigoplus_{i\in I} M_i(P)$. On the other hand, let $x_i \in M_i(P)$ then $sx_i \in PM_i$, for some $s \in R\backslash P$. Let $\overline{x} = \{x_j\}_{j\in I}$ and $x_j = 0$ whenever $j \neq i$. Then $sx_i = s\overline{x} \in PM_i \subseteq P(\bigoplus_{i\in I} M_I)$ and so $x_i \in (\bigoplus_{i\in I} M_i)(P)$, that is, $M_i(P) \subseteq (\bigoplus_{i\in I} M_i)(P)$. Finally we have $\bigoplus_{i\in I} M_i(P) \subseteq (\bigoplus_{i\in I} M_i)(P)$.

We recall that an R-module M is said to be *weakly finitely generated* if for any $P \in Supp_R(M)$, the submodule $M(P)$ of M is proper, and hence by Proposition 3.1, $M(P)$ is a P-prime submodule of M.

Definition 3.8. Let M be a weakly finitely generated R-module. The sets of *associated* and of *supported prime submodules* of M are defined, respectively, as follows:
$Ass_P(M) = \{M(P)|P \in Ass_R(M)\}$ and $Supp_P(M) = \{M(P)|P \in Supp_R(M)\}$.

Corollary 3.9. *Let R be a ring which satisfies the ascending chain condition on semiprime ideals. Then every multiplication R-module is finitely generated.*

Proof. See [4, Corollary 3.9].

Remarks. (1) Let the R-module M be either finitely generated or multiplication then, by Theorem 3.6, it is weakly finitely generated.

(2) If R is a Noetherian ring then, by Corollary 3.9, every multiplication R-module is finitely generated.

(3) If M is a Noetherian R-module then the ring $T = R/Ann_R(M)$ is a Noetherian ring. Because M is a faithful $R/Ann_R(M)$-module and if $R' = R/Ann_R(M)$ then M is also Noetherian as an R'-module. Let $M = R'x_1 + ... + R'x_n$ then R' can be embedded in $M^n = \bigoplus_{i=1}^n M_i \quad (M_i = M)$ by the map $a \mapsto (ax_1, ..., ax_n)$. Since by [7, Corollary 7.21], M^n is Noetherian, then so is $R' = R/Ann_R(M)$.

(4) By the remark between Exercises 7.27 and 7.28 of [7], M is a $T = R/Ann_R(M)$-module if and only if it is an R-module via the rule:

$$(r + Ann_R(M))x = rx, \quad \forall r \in R,\ x \in M.$$

If $\overline{P} \in Ass_T(M)$ then $\overline{P} = P/Ann_R(M)$, where P is a prime ideal of R. Also $\overline{P} = Ann_T(x)$ for some $x \in M$, $x \neq 0$. If $\overline{r} = r + Ann_R(M) \in \overline{P}$ then $\overline{r}x = 0$ and so $rx = 0$, where $r \in P$. Hence $P = Ann_R(M)$ and we have

$$Ass_T(M) = \{P/Ann_R(M) \mid P \in Ass_R(M)\}.$$

Lemma 3.10. *Let M be a Noetherian R-module. Then*
(1) $Ass_P(M)$ is a finite set.
(2) $M = 0$ if and only if $Ass_P(M) = \emptyset$.

Proof. (1) By the definition, $Ass_P(M) = \{M(P) \mid P \in Ass_R(M)\}$. Since M is a Noetherian R-module, by Lemma 2.7, $Ass_R(M)$ is a finite set. Therefore $Ass_P(M)$ is also finite.
(2) If $M = 0$ then, by Lemma 2.2, $M_P = 0$ for every $P \in Spec(R)$. This implies $Supp_R(M) = \emptyset$ and since $Ass_R(M) \subseteq Supp_R(M)$, we have $Ass_R(M) = \emptyset$. Therefore $Ass_P(M) = \emptyset$. On the other hand, if $Ass_P(M) = \emptyset$ then $Ass_R(M) = \emptyset$. Now the ring $T = R/Ann_R(M)$ is Noetherian and $Ass_T(M) = \{P/Ann_R(M) \mid P \in Ass_R(M)\}$. Therefore $Ass_T(M) = \emptyset$. Finally, by Proposition 2.6, $M = 0$ as an R (and as well as an $R/Ann_R(M)$)-module.

Theorem 3.11. *Let M be an R-module of finite length. Then*
(1) $Ass_P(M) = Supp_P(M)$ and this is a finite set.
(2) If M is multiplication, then $Ass_P(M) = Max(M) = Spec(M)$, where $Max(M)$ is the set of all maximal submodules of M.

Proof. (1) Since M is a module of finite length it is both Artinian and Noetherian. Since M is a Noetherian R-module the ring $T = R/Ann_R(M)$ is a Noetherian ring.

Hence M is a T-module and

$$Ass_T(M) = \{P/Ann_R(M) \mid P \in Ass_R(M)\}.$$

Also since M is Artinian and finitely generated as an R-module, by a proof similar to that given in Remark(3) after Corollary 3.9, the ring T is Artinian and we can show that

$$Supp_T(M) = \{P/Ann_R(M) \mid P \in Supp_R(M)\}.$$

If we can prove that $Supp_T(M) = Ass_T(M)$, then it is clear that $Supp_R(M) = Ass_R(M)$. By Lemma 2.7, $Ass_T(M)$ (as well as $Ass_R(M)$) is finite. By [1, Corollary 1.6.10], $Ass_T(M) = Supp_T(M)$ and hence $Ass_R(M) = Supp_R(M)$. Now by Definition 3.8 we have $Ass_P(M) = Supp_P(M)$ and this is a finite set.
(2) Let K be a maximal submodule of M, then K is a prime submodule with $P = (K : M)$ a maximal ideal of R. Note that $Ann_R(M) \subseteq P$ so, by Lemma 2.5, $P \in Supp_R(M)$ $(= Ass_R(M))$. Since M is a multiplication module, $K = (K : M)M = PM$. We know that $PM \subseteq M(P)$, and if $x \in M(P)$ then $sx \in PM$ for some $s \in R \backslash P$. But $K = PM$ is prime, so $x \in PM$ or $s \in (PM : M) = P$. This implies $x \in PM$ and hence $M(P) = PM = K$. Therefore $Max(M) \subseteq Ass_P(M)$.
Here we prove that if M is a multiplication module and K is a proper submodule of M such that $(K : M) = Q$ is a maximal ideal of R, then K is a maximal submodule of M. Because if L is a submodule of M such that $K \subset L \subset M$ then $(K : M) \subset (L : M) \subset (M : M) = R$. But this contradicts the maximality of $(K : M) = Q$. Therefore K is a maximal submodule of M. Consider an element $M(P) \in Ass_P(M)$, so $P \in Ass_R(M)$ and clearly $P \supseteq Ann_R(M)$. Hence $P/Ann_R(M)$ is a maximal ideal of the Artinian ring $R/Ann_R(M)$ and therefore P is a maximal ideal of R. We also have $M(P) = PM$ and $(M(P) : M) = (PM : M) = P$. Thus $M(P)$ is a maximal submodule of M and hence $Ass_P(M) \subseteq Max(M)$. Finallly, we show that $Ass_P(M) = Max(M) = Spec(M)$. Let $K \in Spec(M)$ with $P = (K : M)$ a prime ideal of R. Clearly $P \supseteq Ann_R(M)$ and so $P \in Supp_R(M) = Ass_R(M)$, hence

$M(P) \in Ass_P(M)$. Now $P = (K : M)$ implies that $PM = (K : M)M = K$ and also $PM = M(P)$. Therefore $K = M(P) \in Ass_P(M)$ and the proof is complete.

Theorem 3.12. *Let M be a multiplication R-module. Then the following statements are equivalent:*

(1) M is an Artinian R-module.

(2) M is a Noetherian R-module and $Ass_P(M) \subseteq Max(M)$.

Proof. (1) $\Rightarrow$ (2). By [4, Corollary 2.9], Artinian multiplication modules are cyclic. Hence $M = Rx$ for some $x \in M$. Since M is an Artinian finitely generated module, the ring $R/Ann_R(M)$ is an Artinian ring and therefore it is Noetherian. But then M is a Noetherian $R/Ann_R(M)$-module and hence a Noetherain R-module. Next we show that $Ass_P(M) \subseteq Max(M)$. Let $M(P) \in Ass_P(M)$, where $P \in Ass_R(M)$. Since $R/Ann_R(M)$ is Artinian, $P/Ann_R(M)$ is a maximal ideal of $R/Ann_R(M)$ and hence P is a maximal ideal of R. Since M is multiplication, $M(P) = PM$ and $(M(P) : M) = (PM : M) = P$ and also $M(P) = PM$ is a maximal submodule of M, that is, $M(P) \in Max(M)$. Therefore $Ass_P(M) \subseteq Max(M)$.

(2)$\Rightarrow$ (1). Let $P \in Ass_R(M)$, then $M(P) \in Ass_P(M) \subseteq Max(M)$. By Lemma 3.4(2), $M(P)$ is a P-prime submodule of M, that is, $(M(P) : M) = P$. Since $M(P)$ is a maximal submodule of M, by [5, Proposition 3 page 63], P is a maximal ideal of R. Since M is Noetherian, the ring $R/Ann_R(M)$ is also Noetherian. Now by [1, Theorem 1.6.9], $L_{R/Ann_R(M)}(M)$ is finite, that is, $L_R(M)$ is finite. Therefore M is an Artinian module.

Theorem 3.13. *Let R be a local ring and M, N be non-zero finitely generated R-modules. Then for any $P \in Spec(R)$*

$$M(P) \otimes_R N(P) = (M \otimes_R N)(P).$$

Proof. With the assumptions of the theorem; $M \otimes_R N = 0$ if and only if $M = 0$ or $N = 0$. Also since M and N are finitely generated R-modules so is $M \otimes_R N$. Let $\Sigma_{i=1}^n x_i \otimes y_i \in M(P) \otimes_R N(P)$, where $x_i \in M(P)$ and $y_i \in N(P)$. Hence for each

$1 \leq i \leq n$ there exist $r_i, s_i \in R\backslash P$ such that $r_i x_i \in PM$ and $s_i y_i \in PN$. Let $\gamma = r_1 ... r_n s_1 ... s_n$, then $\gamma \in R\backslash P$ and $\gamma \Sigma_{i=1}^{n} x_i \otimes y_i = \Sigma_{i=1}^{n} \gamma x_i \otimes y_i \in PM \otimes_R N$.By the way that $M \otimes_R N$ is made an R-module, we have $P(M \otimes_R N) = (PM \otimes_R N) = (M \otimes_R PN)$. Therefore $\gamma \Sigma_{i=1}^{n} x_i \otimes y_i \in P(M \otimes_R N)$ and so $\Sigma_{i=1}^{n} x_i \otimes y_i \in (M \otimes_R N)(P)$. Next let $\Sigma_{i=1}^{n} x_i' \otimes y_i' \in (M \otimes_R N)(P)$. Then $r\Sigma_{i=1}^{n} x_i' \otimes y_i' \in P(M \otimes_R N)$ for some $r \in R\backslash P$. We show that $x_i' \in M(P)$ and $y_i' \in N(P)$ for each $1 \leq i \leq n$. Assume the contrary, and let there exists $1 \leq j \leq n$ such that $x_j' \notin M(P)$ or $y_j' \notin N(P)$. Therefore $s' x_j' \notin PM$ or $s'' y_j' \notin PN$ for any $s', s'' \in R\backslash P$. Suppose $s' x_j' \notin PM$ for any $s' \in R\backslash P$, then $r\Sigma_{i=1}^{n} x_i' \otimes y_i' = \Sigma_{i=1}^{n} r x_i' \otimes y_i' \in P(M \otimes_R N) = PM \otimes_R N$ and so $r x_j' \in PM$, which is a contradiction.We conclude that $\Sigma_{i=1}^{n} x_i' \otimes y_i' \in M(P) \otimes_R N(P)$.

Lemma 3.14. *Let M, M' and M'' be weakly finitely generated R-modules and the sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be exact. Then*

$$Supp_P(M) = Supp_P(M') \cup Supp_P(M'').$$

Proof. Let $M(P) \in Supp_P(M)\backslash Supp_P(M')$, then $P \in Supp_R(M)\backslash Supp_R(M')$. We know that the sequence $0 \longrightarrow M_P' \longrightarrow M_P \longrightarrow M_P'' \longrightarrow 0$ is also exact. Hence $M_P' = 0$ and the map $M_P \longrightarrow M_P''$ is injective, as well as, surjective and so it is an isomorphism. But by the assumption, $M_P \neq 0$ and hence $M_P'' \neq 0$. Thus we have $P \in Supp_R(M'')$. Therefore $M(P) \in Supp_P(M'')$ and $Supp_P(M) \subseteq Supp_P(M') \bigcup Supp_P(M'')$. On the other hand, since M_P' is isomorphic to a submodule of M_P it follows that $Supp_R(M') \subseteq Supp_R(M)$ and hence $Supp_P(M') \subseteq Supp_P(M)$. If $M_P = 0$, then $M_P'' = 0$ (because the map $M_P \longrightarrow M_P''$ is surjective). Equivalently, if $M_P'' \neq 0$ then $M_P \neq 0$. Thus $Supp_R(M'') \subseteq Supp_R(M)$ and so $Supp_P(M'') \subseteq Supp_P(M)$. Therefore $Supp_P(M') \bigcup Supp_P(M'') \subseteq Supp_P(M)$.

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