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A collocation method for the solution of nonlinear one-dimensional parabolic equations

J. Rashidinia[a,b,1], M. Ghasemi[a], R. Jalilian[c]

[a]School of Mathematics, Iran University of Science & Technology Narmak, Tehran 16844, Iran

[b]Department of Mathematics, Islamic Azad University, Central Branch, Tehran, Iran

[c]Department of Mathematics, Ilam University, PO Box 69315-516, Ilam, Iran

Abstract

In this paper, we develop a collocation method based on cubic B-spline to the solution of nonlinear parabolic equation $\varepsilon u_{xx} = a(x,t)u_t + \phi(x,t,u,u_x)$ subject to appropriate initial, and Dirichlet boundary conditions, where $\varepsilon > 0$ is a small constant. We developed a new two-level three-point scheme of order $O(k^2 + h^2)$. The convergence analysis of the method is proved. Numerical results are given to illustrate the efficiency of our method computationally.

Keywords: Cubic B-spline method; Nonlinear parabolic equation; Singularly perturbed; Convection-Diffusion equation; Burgers equation; Convergence analysis.

1 Introduction

Consider the following one-space dimensional nonlinear parabolic equation,

$$\varepsilon u_{xx} = a(x,t)u_t + \phi(x,t,u,u_x) \qquad (x,t) \in (0,1) \times (0,T], \tag{1}$$

subject to the initial condition,

$$u(x,0) = f(x), \quad 0 \le x \le 1, \tag{2}$$

[1]Corresponding Author. E-mail Address: rashidinia@iust.ac.ir

and boundary conditions,

$$u(0,t)=\alpha(t), \qquad u(1,t)=\beta(t), \;\; t\geq 0, \tag{3}$$

where ϕ is smooth continues nonlinear function and $a(x,t)>a^{*}>0$. Suppose that $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial u_x}$ exist and $\frac{\partial \phi}{\partial u}>0$ and also $\phi'(z),\;\; z\in \Re^4$ exists and bounded.

The numerical solution of nonlinear parabolic equation both in Cartesian and polar coordinates are of great importance in problems of viscous fluid flow and heat transfer. For example, Burger's equation, which arises in the theory of shock waves, in turbulence problems and in continuous stochastic processes. It shows a structure roughly similar to that of Navier-Stokes equations due to the form of the nonlinear convection term and the occurrence of the Reynolds number term. Jain et al. [6] and [7] have developed the finite difference method of order $O(k^2+h^4)$ for solving (1) both in scalar and vector form. Mohanty [11] has discussed difference methods of $O(k^2+h^4)$ for Burger's equation and the heat conduction equation in polar coordinates, and Mohanty and Jain [12] have developed the single cell finite difference approximations of $O(kh^2+h^4)$ for $\partial u/\partial x$ for one space dimensional non-linear parabolic equations. The methods described in [6], [7], [11] and [12] are of high orders; However, those methods are not directly applicable to solve singular parabolic equations. Rigal [16] have proposed two-level high-order implicit schemes, which were shown to be accurate and cost effective. In [3] and [9] high-order unconditionally stable difference schemes for variable coefficients parabolic equations were discussed. Douglas and Jones [4] have used predictor-corrector technique to solve non-linear parabolic equations. Second order convergent numerical methods for the solution of non-linear parabolic equation (1) have been analyzed by Reynolds [1] and Jacques [5]. Recently, Mohanty et al. [10], [13], [14] and Urvashi Arora et al. [2] have developed some scheme based on finite difference method. Kadalbajoo et al. [8] developed a B-spline collocation method on a non-uniform mesh for singularly perturbed one-dimensional problems and obtained an unconditionally stable method of order $O((\Delta x)^2+\Delta t)$ for time-dependent linear convection-diffusion problems.

In this paper we have developed two-level implicit difference scheme by using cubic B-spline function for the solution of singularly perturbed one-dimensional time-dependent non-linear convection diffusion problem (1). This paper is arranged as follows. In section 2, we present a finite difference approximation to discretize the equation (1) and obtain the convergence of method in time variable. In section 3 we applied cubic B-spline collocation method to solve the ordinary differential equations obtained in each time level. The uniform convergence of the method is proved in Section 4. In Section 5, numerical experiments are conducted to demonstrate the viability and the efficiency of the proposed method computationally.

2 Temporal discretization

Let us consider a uniform mesh Δ with the grid points $\lambda_{i,j}$ to discretize the region $\Omega = [0,1] \times (0,T]$. Each $\lambda_{i,j}$ is the vertices of the grid point (x_i, t_j) where $x_i = ih$, $i = 0,1,2,...,N$ and $t_j = jk$, $k = 0,1,2,...$ and h and k are mesh sizes in the space and time directions respectively.

First we use the following finite difference approximation to discretize the time variable with uniform step size k,

$$u_t^n \cong \frac{\delta_t}{k(1-\gamma\delta_t)} u^n, \qquad n \geq 0 \ , \ \gamma \neq 1, \tag{4}$$

where n is the step number in t direction and $\delta_t u^n = u^n - u^{n-1}$, $u^n = u(x, t_n)$ and $u^0 = u(x,0) = f(x), \quad (0 \leq x \leq 1)$.

Substituting the above approximation into equation (1) and discretizing in time variable we have:

$$\varepsilon u_{xx}^n = a(x,t_n) \frac{\delta_t}{k(1-\gamma\delta_t)} u^n + \phi(x, t_n, u^n, u_x^n), \tag{5}$$

thus we have,

$$\varepsilon k(1-\gamma\delta_t) u_{xx}^n = a(x,t_n)\delta_t u^n + k(1-\gamma\delta_t)\phi(x, t_n, u^n, u_x^n). \tag{6}$$

Now by simplifying we can write equation (6) in the following form

$$-\varepsilon u^*_{xx} + p^*(x)u^* + \phi^*(x, u^*, u^*_x) = q^*(x), \tag{7}$$

where $u^* \equiv u^n$ and,

$$\begin{cases} p^*(x) = \frac{a(x,t_n)}{(1-\gamma)k}, \\ q^*(x) = \frac{\varepsilon\gamma}{1-\gamma}u^{n-1}_{xx} + \frac{a(x,t_n)}{(1-\gamma)k}u^{n-1} - \frac{\gamma}{1-\gamma}\phi(x, t_n, u^{n-1}, u^{n-1}_x) \\ \phi^*(x, u^*, u^*_x) = \phi(x, t_n, u^n, u^n_x). \end{cases}$$

with boundary conditions ,

$$u^*(0) = \alpha(t_n), \quad u^*(1) = \beta(t_n). \tag{8}$$

Thus, now in each time level we have a nonlinear ordinary differential equation in the form of (7) with the boundary conditions (8) which can be solved by using B-spline collocation method.

Theorem 2.1 :

The above time disretization process that we use to disretize equation (1) in time variable is of the second order convergence.

Proof :

Let $u(t_i)$ be the exact solution and u^i the approximate solution of the problem (1) at the ith level time and also suppose that $e_i = u^i - u(t_i)$ be the local truncation error in (7), then using equation (4) and replacing $\gamma = \frac{1}{2}$ it can be easily proved that,

$$|e_i| \leq C_i k^3, \tag{9}$$

where C_i is some finite constant.

Let E_{n+1} be the global error in time discretizing process then the global error in $(n+1)th$ level is $E_{n+1} = \sum_{i=1}^{n} e_i, \quad (t \leq T/n)$ thus with the help of (9) we have :

$$|E_{n+1}| \;=\; |\sum_{i=1}^{n} e_i| \;\leq\; \sum_{i=1}^{n} |e_i| \;\leq\; \sum_{i=1}^{n} C_i k^3 \;\leq\; nC^{'}k^3 \;\leq\; C^{'}n(T/n)k^2 = Ck^2,$$

where $C = C^{'}T$, which gives the second order convergent of the method in time direction easily.

3 B-spline collocation method

In this section we use B-spline collocation method to solve (7) with the boundary conditions (8) in each time level. Let $\Delta^\star = \{a = x_0 < x_1 < ... < x_N = b\}$ be the partition in $[a, b]$. B-splines are the unique nonzero splines having smallest compact support with nodes at $x_0 < x_1 < ... < x_{N-1} < x_N$. We define the cubic B-splines for $i = -1, 0, ..., N+1$ by the following relation [15].

$$B_{3,i} = \frac{1}{h^3}\begin{cases} (x - x_{i-2})^3, & x \in [x_{i-2}, x_{i-1}], \\ h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3, & x \in [x_{i-1}, x_i], \\ h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3, & x \in [x_i, x_{i+1}], \\ (x_{i+2} - x)^3, & x \in [x_{i+1}, x_{i+2}], \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

It can be easily seen that the set of functions $\Omega = \{B_{-1}(x), B_0(x), ..., B_{N+1}(x)\}$ are linearly independent on $[a, b]$ thus $\Theta = \text{Span}(\Omega)$ is a subspace of $C^2[a, b]$, and Θ has N+3 dimension. Let us consider that $S^*(x) \in \Theta$ be the B-spline approximation to the exact solution of problem (7), thus we can write $S^*(x)$ in the following form:

$$S^*(x) = \sum_{i=-1}^{N+1} c_i^* B_i(x), \tag{11}$$

where c_i^* are unknown real coefficients and $B_i(x)$ are cubic B-spline functions. By forcing $S^*(x)$ to satisfy the collocation equations plus the boundary conditions, we have:

$$LS^*(x_i) = q^*(x_i), \quad 0 \le i \le N \tag{12}$$

$$S^*(x_0) = \alpha(t_n) \quad , \quad S^*(x_N) = \beta(t_n)$$

where $Lu^* \equiv -\varepsilon u^*_{xx} + p^*(x)u^* + \phi^*(x, u^*, u^*_x)$.

Substituting (11) into (12) and using the properties of B-spline functions we have:

$$\frac{-6\varepsilon}{h^2}(c^*_{i-1} - 2c^*_i + c^*_{i+1}) + p^*(x_i)(c^*_{i-1} + 4c^*_i + c^*_{i+1}) + \phi^*(x_i, (c^*_{i-1} + 4c^*_i + c^*_{i+1}), \frac{3}{h}(c^*_{i+1} - c^*_{i-1})) = q^*(x_i),$$

$$0 \le i \le N,$$

simplifying the above relation leads to the following system of $(N-1)$ non-linear equations in $(N+3)$ unknowns:

$$(-6\varepsilon+h^2p_i^*)c_{i-1}^*+(12\varepsilon+4h^2p_i^*)c_i^*+(-6\varepsilon+h^2p_i^*)c_{i+1}^*+h^2\phi_i^* = h^2q_i^*, \qquad 1 \le i \le N-1, \tag{13}$$

where $p_i^* = p^*(x_i), q_i^* = q^*(x_i)$ and $\phi_i^* = \phi^*(x_i, (c_{i-1}^* + 4c_i^* + c_{i+1}^*), \frac{3}{h}(c_{i+1}^* - c_{i-1}^*))$.
To be able to solving the system, and obtaining a unique solution to $C^* = (c_{-1}^*, c_0^*, ..., c_N^*, c_{N+1}^*)$ we need to obtain four extra equations to be associated with (13). This can be done by using the boundary conditions and then eliminating c_{-1}^* and c_{N+1}^*.
Using the left boundary condition we have,

$$u(0,t_n) = S^*(0) = \alpha(t_n) = c_{-1}^* + 4c_0^* + c_1^*,$$

Now eliminating c_{-1}^* from the above equation and substituting into equation (13) for $i = 0$ we have,

$$(-6\varepsilon + h^2p_0^*)(\alpha(t) - 4c_0^* - c_1^*) + (12\varepsilon + 4h^2p_0^*)c_0^* + (-6\varepsilon + h^2p_0^*)c_1^* + h^2\phi_0^* = h^2q_0^*.$$

Simplifying the above equation we obtain the following relation,

$$36\varepsilon c_0^* + h^2\phi_0^* = h^2q_0^* + (6\varepsilon - h^2p_0^*)\alpha(t_n), \qquad \text{for i=0} \tag{14}$$

where

$$\phi_0^* = \phi^*(x_0, \alpha(t_n), \frac{3}{h}(c_1^* - c_{-1}^*)) = \phi^*(x_0, \alpha(t_n), \frac{3}{h}(4c_0^* + 2c_1^* - \alpha(t_n)).$$

Similarly by using the right boundary condition we have,

$$u(1,t_n) = S^*(1) = \beta(t_n) = c_{N-1}^* + 4c_N^* + c_{N+1}^*,$$

and eliminating c_{N+1}^* from the last equation (13) for $i = N$, we find

$$36\varepsilon c_N^* + h^2\phi_N^* = h^2q_N^* + (6\varepsilon - h^2p_N^*)\beta(t), \qquad \text{for i=N,} \tag{15}$$

where

$$\phi_N^* = \phi^*(x_N, u(x_N), u_x(x_N)) = \phi^*(x_N, \beta(t), \frac{3}{h}(c_{N+1}^* - c_{N-1}^*)) = \phi^*(x_N, \beta(t), -\frac{3}{h}(4c_N^* + 2c_{N-1}^* - \beta(t)).$$

Equations (14), (15) can be associated with (13) which gives the following $(N+1)$ in $(N+1)$ non-linear system ,

$$AC^* + h^2 B^* = h^2 Q^*, \tag{16}$$

where ,

$$A = \begin{pmatrix} 36\varepsilon & 0 & 0 & & \\ -6\varepsilon + h^2 p_1^* & 12\varepsilon + 4h^2 p_1^* & -6\varepsilon + h^2 p_1^* & & \\ & \ddots & \ddots & \ddots & \\ & & -6\varepsilon + h^2 p_1^* & 12\varepsilon + 4h^2 p_1^* & -6\varepsilon + h^2 p_1^* \\ & & 0 & 0 & 36\varepsilon \end{pmatrix}, \tag{17}$$

and ,

$$B^* = \begin{pmatrix} \phi_0^* \\ \phi_1^* \\ \vdots \\ \phi_N^* \end{pmatrix}, \quad Q^* = \begin{pmatrix} q_0^* \\ q_1^* \\ \vdots \\ q_N^* \end{pmatrix}, \quad C^* = \begin{pmatrix} c_0^* \\ c_1^* \\ \vdots \\ c_N^* \end{pmatrix}. \tag{18}$$

4 Convergence analysis

Let $u^*(x)$ be the exact solution of the equation (7) with the boundary conditions (8) and also $S^*(x) = \sum_{i=-1}^{N+1} c_i^* B_i(x)$ be the B-spline collocation approximation to $u^*(x)$. Due to round off errors in computations we assume that $\hat{S}(x)$ be the computed spline for $S^*(x)$ so that $\hat{S}(x) = \sum_{i=-1}^{N+1} \hat{c}_i B_i(x)$ where $\hat{C} = (\hat{c}_0, \hat{c}_1, ..., \hat{c}_{N-1}, \hat{c}_N)$. To estimate the error $\| u^*(x) - \hat{S}(x) \|_\infty$ we must estimate the errors $\| u^*(x) - S^*(x) \|_\infty$ and $\| \hat{S}(x) - S^*(x) \|_\infty$ separately.

Following (16) for $\hat{S}(x)$ we have ,

$$A\hat{C} + h^2 \hat{B} = h^2 \hat{Q}, \tag{19}$$

where $\hat{B} = [\hat{\phi}_0, \hat{\phi}_1, ..., \hat{\phi}_N]^T, \hat{Q} = (\hat{q}_0, \hat{q}_1, ..., \hat{q}_N)^T$ and $\hat{\phi}_i = \phi^*(x_i, (\hat{c}_{i-1}+4\hat{c}_i+\hat{c}_{i+1}), \frac{3}{h}(\hat{c}_{i+1}-\hat{c}_{i-1}))$.

Substracting (19) and (16) we have ,

$$A(C^* - \hat{C}) + h^2(B^* - \hat{B}) = h^2(Q^* - \hat{Q}). \tag{20}$$

First we need to recall some theorems.

Theorem 4.1

Suppose that $f(x) \in C^4[0,1]$ and $|f^{(4)}(x)| \leq L$, $\forall x \in [0,1]$ and $\Delta = \{0 = x_0 < x_1 < ... < x_N = 1\}$ be the equally spaced partition of $[0,1]$ with step size h. If $S_\Delta(x)$ be the unique spline function interpolate $f(x)$ at nodes $x_0, ..., x_N \in \Delta$, then there exist a constant $\lambda_j \leq 2$ such that $\forall x \in [0,1]$,

$$\|f^{(j)}(x) - S_\Delta^{(j)}(x)\| \leq \lambda_j L h^{4-j}, \qquad j = 0, 1, 2, 3. \tag{21}$$

where $\|.\|$ represents the ∞-norm.

Proof : For the proof see Stoer and Bulirsch [18] (pp. 105).

Now we want to find a bound on $\|Q^* - \hat{Q}\|_\infty$ first. Applying (12) we have ,

$$| q^*(x_i) - \hat{q}(x_i) | = | LS^*(x_i) - L\hat{S}(x_i) | =$$

$$| -\varepsilon S^{*''}(x_i) + \varepsilon \hat{S}''(x_i) + p^*(x_i)S^*(x_i) - p^*(x_i)\hat{S}(x_i) + \phi^*(x_i, S^*(x_i), S^{*'}(x_i)) - \phi^*(x_i, \hat{S}(x_i), \hat{S}'(x_i)) |,$$

thus we have ,

$$|q^*(x_i) - \hat{q}(x_i)| \leq$$

$$| -\varepsilon S^{*''}(x_i) + \varepsilon \hat{S}''(x_i) | + | p^*(x_i) || S^*(x_i) - \hat{S}(x_i) | + | \phi^*(x_i, S^*(x_i), S^{*'}(x_i)) - \phi^*(x_i, \hat{S}(x_i), \hat{S}'(x_i)) | . \tag{22}$$

From (22) and by following Theorem 4.1 and [17] (page 218) we obtain ,

$$\begin{aligned} \|Q^* - \hat{Q}\|_\infty &\leq \varepsilon \lambda_2 L h^2 + \|p^*(x)\|_\infty \lambda_0 L h^4 + M(|S^*(x) - \hat{S}(x)| + |S^{*'}(x) - \hat{S}'(x)|) \\ &\leq \varepsilon \lambda_2 L h^2 + \|p^*(x)\|_\infty \lambda_0 L h^4 + ML\lambda_0 h^4 + M\lambda_1 L h^3, \end{aligned} \tag{23}$$

where $\|\phi'(z)\|_\infty \le M$. Thus we can rewrite (23) as follows ,

$$\|Q^* - \hat{Q}\|_\infty \le M_1 h^2, \tag{24}$$

where $M_1 = \varepsilon\lambda_2 L + \|p^*(x)\|_\infty \lambda_0 L h^2 + M L \lambda_0 h^2 + M\lambda_1 L h$.

Also we consider the nonlinear term in left hand side of (20) as follows

$$h^2(B^* - \hat{B}) =$$

$$h^2(\begin{pmatrix} \phi^*(x_0, \alpha(t), \frac{3}{h}(4c_0^* + 2c_1^* - \alpha(t))) \\ \vdots \\ \phi^*(x_i, (c_{i-1}^* + 4c_i^* + c_{i+1}^*), \frac{3}{h}(c_{i+1}^* - c_{i-1}^*)) \\ \vdots \\ \phi^*(x_N, \beta(t), \frac{3}{h}(4c_N^* + 2c_{N-1}^* - \alpha(t))) \end{pmatrix} - \begin{pmatrix} \phi^*(x_0, \alpha(t), \frac{3}{h}(4\hat{c}_0 + 2\hat{c}_1 - \alpha(t))) \\ \vdots \\ \phi^*(x_i, (\hat{c}_{i-1} + 4\hat{c}_i + \hat{c}_{i+1}), \frac{3}{h}(\hat{c}_{i+1} - \hat{c}_{i-1})) \\ \vdots \\ \phi^*(x_N, \beta(t), \frac{3}{h}(4\hat{c}_N + 2\hat{c}_{N-1} - \alpha(t))) \end{pmatrix}).$$

Applying the mean value theorem we have

$$h^2(B^* - \hat{B}) \quad = \quad [h^2(\frac{\partial \phi^*}{\partial u})(\xi_1) J + h^2(\frac{\partial \phi^*}{\partial u_x})(\xi_2) D](C^* - \hat{C}),$$

where ξ_1 and ξ_2 are in $(0, 1)$ and J and D are given by the following $(N+1)\times(N+1)$ matrixes

$$J = \begin{pmatrix} 0 & 0 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} \frac{12}{h} & \frac{6}{h} & & & \\ \frac{-3}{h} & 0 & \frac{3}{h} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{-3}{h} & 0 & \frac{3}{h} \\ & & & \frac{6}{h} & \frac{12}{h} \end{pmatrix}.$$

Substituting (25) into (20) and simplifying we have ,

$$R(C^* - \hat{C}) = h^2(Q^* - \hat{Q}), \tag{25}$$

where $R = A + h^2(\frac{\partial \phi^*}{\partial u})(\xi_1) J + h^2(\frac{\partial \phi^*}{\partial u_x})(\xi_2) D$.

It is obvious that the matrix R is a strictly diagonally dominant matrix and hence it is nonsingular, thus from (25) we have,

$$(C^* - \hat{C}) = h^2 R^{-1}(Q^* - \hat{Q}). \tag{26}$$

Taking the infinity norm and then by using (24) we find,

$$\|(C^* - \hat{C})\|_\infty \leq h^2\|R^{-1}\|_\infty\|(Q^* - \hat{Q})\|_\infty \leq h^2\|R^{-1}\|_\infty(h^2M_1) = \|R^{-1}\|_\infty M_1h^4. \quad (27)$$

Now suppose that χ_i is the summation of the ith row of the matrix $R = [r_{ij}]_{(N+1)\times(N+1)}$, then we have:

$$\chi_0 = \sum_{j=0}^{N} r_{0,j} = 36\varepsilon + 18h\frac{\partial\phi^*}{\partial u'}, \qquad i = 0, \quad (28)$$

$$\chi_i = \sum_{j=0}^{N} r_{i,j} = 6h^2p_i^* + 6h^2\frac{\partial\phi^*}{\partial u}, \qquad i = 1, \cdots, N-1, \quad (29)$$

$$\chi_N = \sum_{j=0}^{N} r_{N,j} = 36\varepsilon + 18h\frac{\partial\phi^*}{\partial u'}, \qquad i = N. \quad (30)$$

From the theory of matrices we have :

$$\sum_{i=0}^{N} r_{k,i}^{-1}\chi_i = 1 \quad , \quad k = 0, ..., N, \quad (31)$$

where $r_{k,i}^{-1}$ are the elements of R^{-1}. Therefor,

$$\|R^{-1}\|_\infty = \sum_{i=0}^{N} |r_{k,i}^{-1}| \leq \frac{1}{min_{0\leq i\leq N}\chi_i} = \frac{1}{h^2\mu_l} \leq \frac{1}{h^2 \mid \mu_l \mid}, \quad (32)$$

where l is some index between 0 and N. Now substituting (32) into (27) and simplifying we have ,

$$\|C^* - \hat{C}\|_\infty \leq \frac{M_1h^4}{h^2 \mid \mu_l \mid} \leq M_2h^2, \quad (33)$$

where M_2 is some finite constant.

Now we will be able to prove the convergence of our presented method. We recall a following lemma first,

lemma 4.2

The B-splines $\{B_{-1}, B_0, ..., B_{N+1}\}$ are satisfies the following inequality :

$$\mid \sum_{i=-1}^{N+1} B_i(x)\mid \leq 10, \; , \quad (0 \leq x \leq 1). \quad (34)$$

Proof : For proof see [8].

Now observe that we have :

$$S^*(x) - \hat{S}(x) = \sum_{i=-1}^{N+1} (c_i^* - \hat{c}_i) B_i(x), \tag{35}$$

thus taking the infinity norm and using (33) and (34) we get ,

$$\|S^*(x) - \hat{S}(x)\|_\infty = \| \sum_{i=-1}^{N+1} (c_i^* - \hat{c}_i) B_i(x) \|_\infty \leq \|(c_i^* - \hat{c}_i)\|_\infty \mid \sum_{i=-1}^{N+1} B_i(x) \mid \leq 10M_2h^2, \tag{36}$$

Theorem 4.3

Let $u^*(x)$ be the exact solution of (7) with Boundary conditions (8) and let $S^*(x)$ be the B-spline collocation approximation to $u^*(x)$ then the method has second order convergence

$$\|u^*(x) - S^*(x)\| \leq \omega h^2, \tag{37}$$

where $\omega = \lambda_0 L h^2 + 10M_2$ is some finite constant.

Proof :

From theorem 4.1 we have :

$$\|u^*(x) - S^*(x)\| \leq \lambda_0 L h^4, \tag{38}$$

thus substituting from (36) and (38) we have ,

$$\|u^*(x) - S^*(x)\| \leq \|u^*(x) - \hat{S}(x)\| + \|\hat{S}(x) - S^*(x)\| \leq \lambda_0 L h^4 + 10M_2h^2 = \omega h^2,$$

where $\omega = \lambda_0 L h^2 + 10M_2$.

We suppose that $u(x,t)$ be the solution of equation (1) and $U(x,t)$ be the approximate solution by our presented method then we have ,

$$\|u(x,t^n) - U(x,t^n)\| \leq \rho(k^2 + h^2), \quad (\rho \text{ is some finite constant}), \tag{39}$$

thus the order of convergence of our process is $O(k^2 + h^2)$.

5 Numerical illustrations

In order to test the viability of our method based on B-spline collocation, and to demonstrate it's convergence computationally we consider some test problems of partial differential equations. We solve these problems for various values of h and ϵ. The computed results are compared with the theoretical results furthermore we compare our method with some other existing methods to demonstrate the efficiency and performance of our method computationally. The results are shown in tables 1-3. All programs are run in Mathematica 5.1.

Example 1.

Consider the following problem from [2] and [10]

$$\epsilon\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad , \tag{40}$$

$$u(x,0) = sin(\pi x)$$

$$u(0,t) = 0 \quad , \quad u(1,t) = 0$$

with the exact solution $u(x,t) = e^{-\epsilon\pi^2 t}\sin(\pi x)$. The root mean square (RMS) errors for u at $t = 1.0$ and with $k = \frac{1.6}{(n+1)^2}$ are obtained for various values of ϵ and h. we compare the results with the results in [2] and [10]. The results are shown in Table 1

Table 1 :

RMS errors in solutions of problem 1 by :

Our method

ϵ / $N+1$	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-10}
08	4.132(-4)	4.412(-5)	4.446(-6)	4.455(-7)	2.582(-8)	4.451(-12)
16	1.037(-4)	1.116(-5)	1.119(-6)	1.110(-7)	1.117(-8)	1.115(-12)
32	2.594(-5)	2.873(-6)	2.791(-7)	3.006(-8)	2.794(-9)	2.713(-13)
64	5.133(-6)	6.981(-7)	5.307(-8)	6.257(-9)	5.784(-10)	1.241(-13)

Method in [2]

	$\sigma = .75$			$\sigma = 1.25$		
$N+1$	$\nu = 0.01$	$\nu = 0.001$	$\nu = 0.0001$	$\nu = 0.01$	$\nu = 0.001$	$\nu = 0.0001$
08	3.563(-3)	4.275(-4)	4.608(-5)	2.831(-3)	3.262(-4)	3.394(-5)
16	1.897(-3)	2.268(-4)	2.424(-5)	1.311(-3)	1.510(-4)	1.562(-5)
32	1.285(-3)	1.536(-4)	1.614(-5)	8.491(-4)	9.787(-5)	1.011(-5)
64	9.015(-4)	1.077(-4)	1.151(-5)	5.944(-4)	6.852(-5)	7.083(-6)

Method in [10]

	$O(k^2h_l^{-1} + h_l^2)$-method		
$N+1$	$\nu = 0.01$	$\nu = 0.001$	$\nu = 0.0001$
08	5.211(-4)	5.732(-5)	5.788(-6)
16	2.195(-4)	2.432(-5)	2.458(-6)
32	1.342(-4)	1.491(-5)	1.507(-6)
64	9.351(-5)	1.038(-5)	1.050(-6)

Example 2.

We consider the following problem ,

$$\epsilon \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + (\frac{\partial u}{\partial x})^2 - u^2 + f(x,t)$$

$$u(x,0) = \epsilon \; sin(\pi x) cos(\pi x)$$

$$u(0,t) = 0 \quad , \quad u(1,t) = 0$$

The Exact solution of this problem is $u(x,t) = \epsilon \; e^{-\epsilon^2 t} \; sin(\pi x) \; cos(\pi x)$. We solve this problem using B-spline collocation method (13) with $k = \frac{1}{100}$ and various values of h and ϵ. The RMS errors in solutions at $t = 1.0$ are shown in table 2.

Table 2 :

RMS errors in solutions of problem 2 by :

Our method

ϵ	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
$N+1$					
08	3.308(-5)	3.620(-7)	3.6544(-9)	3.658(-11)	3.662(-13)
16	8.167(-6)	8.924(-8)	9.005(-10)	9.013(-12)	9.017(-13)
32	2.034(-6)	2.222(-8)	2.242(-10)	2.245(-12)	2.245(-14)
64	5.080(-7)	5.551(-9)	5.601(-10)	5.606(-12)	5.607(-14)

Example 3. (Burger's Equation)

We consider the following problem from [2] , [10] and [13],

$$\epsilon\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x},\qquad 0<x<1.\qquad t>0, \tag{41}$$

The exact solution is given by

$$u(x,t)=\frac{2\epsilon\pi\sin(\pi x)e^{-\epsilon\pi^2 t}}{2+\cos(\pi x)e^{-\epsilon\pi^2 t}}. \tag{42}$$

where $\epsilon>0$. The RMS errors for u at $t=1.0$ and with $k=\frac{1.6}{(n+1)^2}$ are obtained for various values of $\epsilon>0$ and h. We compared our results with the results in [2] , [10] and [13]. The results are shown in Table 3.

6 Conclusion

A numerical method is developed to solve a one-dimensional singularly perturbed nonlinear parabolic initial-boundary value problem. This process is based on the B-spline collocation method. It has been found that the proposed algorithm gives highly accurate numerical results than the same finite difference schemes. Numerical experiments demonstrated that the method converges faster for the small values of ϵ.

Table 3 :

RMS errors in solutions of problem 3 by :

Our method

ϵ / $N+1$	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
08	1.091(-5)	1.273(-7)	1.4313(-9)	1.441(-11)	1.452(-13)
16	4.725(-6)	6.479(-8)	7.512(-10)	3.624(-12)	6.631(-13)
32	9.201(-7)	1.652(-8)	3.967(-10)	3.752(-12)	5.058(-14)
64	4.459(-7)	9.481(-9)	1.261(-10)	2.182(-12)	2.372(-14)

Method in [2]

	$\sigma = .8$			$\sigma = 1.40$		
$N+1$	$Re = 10^2$	$Re = 10^3$	$Re = 10^4$	$Re = 10^2$	$Re = 10^3$	$Re = 10^4$
08	1.461(-4)	2.350(-6)	2.500(-8)	8.186(-4)	2.079(-5)	3.415(-7)
16	4.703(-5)	6.774(-7)	7.074(-9)	5.206(-4)	1.344(-5)	2.215(-7)
32	2.856(-5)	4.005(-7)	4.174(-9)	3.602(-4)	9.314(-6)	1.533(-7)
64	1.996(-5)	2.797(-7)	2.919(-9)	2.527(-4)	6.533(-6)	1.072(-7)

Method in [10]

$N+1$	$Re = 10^2$	$Re = 10^3$	$Re = 10^4$	$Re = 10^5$	$Re = 10^6$
08	1.720(-5)	2.310(-7)	2.4021(-9)	2.410(-11)	2.412(-13)
16	5.783(-6)	7.523(-8)	7.777(-10)	7.800(-12)	7.804(-14)
32	3.726(-6)	4.831(-8)	4.992(-10)	5.003(-12)	5.009(-14)
64	2.612(-6)	3.386(-8)	3.499(-10)	3.511(-12)	3.518(-14)

Method in [13]

$N+1$	$Re = 10^2$	$Re = 10^3$	$Re = 10^4$
08	1.763(-4)	2.522(-6)	2.622(-8)
16	4.321(-5)	6.388(-7)	6.677(-9)
32	1.066(-5)	1.590(-7)	1.664(-9)

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